Non-zero degree maps between 3-manifolds

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Mars, 2006

A compact, orientable 3-manifold M *dominates* a compact orientable 3-manifold N if there is a non-zero degree proper map $f : M \to N$. When the degree of f is one, we say that M *1-dominates* N.

Question 1. *Does a closed orientable 3-manifold 1-dominates at most finitely many closed, irreducible and orientable 3-manifolds?*

If one allows any degree, a closed orientable 3-manifolds which dominates a 3-manifolds supporting one of the geometries $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R}), Nil$ can dominate infinitely many 3-manifolds. At the moment these are the only known examples,

Question 2. *Let M be a closed orientable 3-manifold. Does M dominate at most finitely many closed, irreducible, orientable 3-manifolds N not supporting the geometries of $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R}), Nil.$?*

The **JSJ-decomposition** of a compact orientable irreducible 3-manifold M is the canonical splitting of M along a finite (possibly empty) collection $\mathcal{T}$ of disjoint and non-parallel, nor boundary-parallel, incompressible, embedded tori into maximal Seifert fibered or atoroidal compact submanifolds.

The components of $M \setminus \mathcal{T}$ are called the **JSJ-pieces** of M.

A compact orientable, irreducible and orientable 3-manifold is called **geometrizable** if the JSJ-pieces of M are either Seifert fibered or admits a hyperbolic structure.

Results of T. Soma and S. Wang-Q. Zhou imply that a closed orientable 3-manifold dominates only finitely many 3-manifolds supporting either a hyperbolic structure or a Seifert geometry $\mathbb{H}^2 \times \mathbb{E}^1$.

$\Rightarrow$ for geometrizable targets Question 2 reduces to the following:

Question 3. *Let M be a closed orientable* 3*-manifold. Does M dominate at most finitely many, closed, orientable, irreducible 3-manifolds N with non-trivial JSJ decomposition?*

There are partial results for Question 3 for sequences of degree 1 maps (Y. Rong and also T. Soma), or when the domain and the target have the same simplicial volume (T. Soma, and also P. Derbez).

P. Derbez solved Question 3 when the domain M is a graph manifold.

A general approach to Question 3 can be divided into two steps:

1. *Finiteness of JSJ-pieces*: show that there is a finite set $\mathcal{HS}(M)$ of compact orientable 3-manifolds such that each JSJ-piece of a 3-manifold N dominated by M belongs to $\mathcal{HS}(M)$.

2. *Finiteness of gluing*: For a given finite set $\mathcal{HS}(M)$ of Seifert manifolds and of complete hyperbolic 3-manifolds with finite volume, there are only finitely many ways of gluing elements in $\mathcal{HS}(M)$ to get closed 3-manifolds dominated by M.

T. Soma proved the finiteness of hyperbolic JSJ-pieces.

We completes the proof of the first step by showing the finiteness of the Seifert fibered JSJ-pieces:

Theorem 1 (Finiteness of JSJ-pieces). *Let M be a closed, orientable, 3-manifold. Then there is a finite set $\mathcal{HS}(M)$ of complete hyperbolic 3-manifolds with finite volume and of Seifert fibered 3-manifolds, such that the JSJ-pieces of any closed, orientable, irreducible, geometrizable 3-manifold N dominated by M belong to $\mathcal{HS}(M)$, provided that N is not supporting the geometries of* $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R}), Nil$.

We prove the finiteness of the gluing when the targets are irreducible, geometrizable, integral homology 3-spheres.

Theorem 2. *Any closed orientable 3-manifold dominates only finitely many geometrizable integral homology 3-spheres.*

Remark that any closed orientable 3-manifold dominates infinitely many rational homology spheres.

A modification of the argument gives:

Corollary 1. *Any compact orientable 3-manifold dominates at most finitely many knot complements in $\mathbb{S}^3$.*

Conjecture[J. Simon] *Given a knot* $k \subset \mathbb{S}^3$*, there are only finitely many knots* $k_i \in \mathbb{S}^3$ *for which there is an epimorphism* $\phi_i : \pi_1(E(k)) \to \pi_1(E(k_i))$*.*

Definition A homomorphism $\phi : \pi_1(E(k)) \to \pi_1(E(k'))$ is non degenerate if it sends the prefered longitude of k to a non-trivial peripheral element of $\pi_1(\partial E(k'))$.

Corollary 2. *Given a knot* $k \subset \mathbb{S}^3$*, there are only finitely many knots* $k_i \subset \mathbb{S}^3$ *for which there is a non degenerate epimorphism* $\phi_i : \pi_1(E(k)) \to \pi_1(E(k_i))$*.*

The finiteness of the Seifert fibered JSJ-pieces follows from a finiteness result for the Thurston norm of all compact 3-manifolds $M_S = M \setminus S)$, where S runs over all incompressible, orientable surfaces (not necessary connected) in M.

This latter result is derived from the finiteness of (a certain version of) "patterned guts" of all these manifolds $M_S = M \setminus S)$. This is a basic principle, which originated from H. Kneser's work.

Suppose S is a closed, incompressible surface in an irreducible 3-manifold M. Then $M_S = M \setminus \mathcal{N}(S)$ is ∂-irreducible and irreducible.

According to Jaco-Shalen-Johannson theory, there is a unique decomposition, up to proper isotopy:

$M_S = (M_S \setminus \text{Seifert pairs}) \cup \text{Seifert pairs}.$

Furthermore the Seifert pairs have unique decompositions, up to proper isotopy:

$\text{Seifert pairs} = (\text{Seifert pairs} \setminus IB_S^-) \cup IB_S^-$

$IB_S^- = I$-bundles over surfaces F with negative Euler characteristic $\chi(F)$ if $\partial F \neq \emptyset$.

Therefore there is a decomposition

$$M_S = (M_S \setminus IB_S^-) \cup_{A_S} IB_S^- = G_S \cup_{A_S} IB_S^-$$

$A_S =$ frontier annuli of IB_S^- in M_S.

Call $G_S = M_S \setminus IB_S^-$ the *guts* of M_S, and the decomposition above the **guts-decomposition** for M_S.

The embeddings of G_S, A_S and IB_S^- are unique up to proper isotopy in M_S.

For each component G of G_S, $G \cap A_S$ is called a ∂-pattern for G_S and the pair $(G, G \cap A_S)$ is called a *patterned guts component* for the surface S.

First step in the proof of the finiteness of the JSJ-pieces

Theorem 3. *Let* M *be a closed, orientable, irreducible 3-manifold. Then there is a finite set* $\mathcal{G}(M)$ *of connected, compact, orientable,* ∂*-patterned 3-manifolds such that for each closed, incompressible (not necessarily connected) surface* $S \subset M$*, all patterned guts components of* $(G_S, G_S \cap A_S)$ *belong to* $\mathcal{G}(M)$*.*

The second step of the proof uses the notion of Thurston norm on the second relative homology group $H_2(X, Y; \mathbb{Z})$ of a compact, orientable 3-manifold X, where $Y \subset \partial X$ is a subsurface.

For a compact orientable surface set $\chi_-(F) = \max\{0, -\chi(F)\}$ if F is connected, otherwise let $\chi_-(F) = \sum \chi_-(F_i)$, where F_i are the components of F.

For an integral class $z \in H_2(M; \mathbb{Z})$ the **Thurston norm** $\|z\|$ of z is defined as $\|z\| = \inf(\max(0, -\chi(F))$ where S runs over all the properly embedded compact orientable surfaces representing the homology class z in $H_2(X, Y; \mathbb{Z})$.

$\| \ \|$ extends to a convex pseudo-norm on $H_2(X, Y; \mathbb{R})$ which is linear on rays through the origin.

Gabai proved that one can replace "embedded surfaces" by "singular surfaces" and still get the same norm.

Let X be a compact, orientable 3-manifold and $Y \subset \partial X$ be a subsurface. For a finite set of elements $\alpha = \{a_1, ..., a_k\}$ of $H_2(X, Y; \mathbb{Z})$ we define:

$TN(\alpha) = \max\{\|a_i\|, i = 1, ..., k\}$.

Then define the **Thurston norm of the pair** (X, Y) to be $TN(X, Y) = \min\{TN(\alpha)\}$ where α runs over all finite generating set of $H_2(X, Y; \mathbb{Z})$

Proposition 1 (Th. norm finiteness). *Let M be an irreducible, closed, orientable 3-manifold. Then $TN(M_S, \partial M_S)$ picks only finitely many values when S runs over all closed, incompressible surfaces embedded in M.*

Proof. For each closed incompressible surface $S \subset M$, consider the guts-decomposition $M_S = G_S \cup_{A_S} IB^-_S$.

The finiteness of patterned guts $\Rightarrow$ finitely many topological types of patterned guts $(G_S, A_S) \Rightarrow$ number of components of A_S is uniformly bounded, for all incompressible surfaces $S \subset M$.

Modify the decomposition so that the gluing annuli between the two parts become separating.

For each component $F \times I$ of IB^-_S choose a curve on the fiber surface F which co-bounds a planar subsurface Q with all boundary components of F.

$|\chi(Q)| \leq |A_S| - 1$ uniformly bounded above for all incompressible surfaces $S \subset M$.

Consider the new decomposition $M = G'_S \cup_{A'_S} IB'^{-}_S$.

G'_S is obtained by gluing to G_S the handlebodies $Q \times I$ along the components of A_S.

IB'^{-}_S is the sub-I-bundle of IB^{-}_S corresponding to the subsurfaces $F - \mathrm{int}(Q)$.

The gluing annuli A'_S are the separating annuli $\partial Q \times I - A_S$.

$\Rightarrow$ only finitely many topological types of ∂-patterned 3-manifolds (G'_S, A'_S) for all incompressible surfaces S in M.

The finiteness for the values of $TN(M_S, \partial M_S)$ follows from:

Lemma 1. *Let $S \subset M$ be a closed incompressible surface, then:* $TN(M_S, \partial M_S) \leq TN(G'_S, \partial G'_S \setminus \mathit{int}A'_S)$.

Proof. Components of A'_S separating $\Rightarrow$

$H_2(M_S, \partial M_S; \mathbb{Z}) \to H_1(A_S, \partial A_S; \mathbb{Z})$ is null

Consider the following natural homomorphisms induced by the inclusion maps:

$\phi : H_2(G'_S, \partial G'_S \setminus \text{int}A'_S; \mathbb{Z}) \to H_2(M_S, \partial M_S; \mathbb{Z})$;

ψ :
$H_2(IB'^{-}_S, \partial IB'^{-}_S \setminus \text{int}A'_S; \mathbb{Z}) \to H_2(M_S, \partial M_S; \mathbb{Z})$.

Show that $\phi + \psi$ is an epimorphism by applying Mayer-Vietoris sequence to the pairs :

$(G'_S, \partial G'_S \setminus \text{int}A'_S)$ and $(IB'^{-}_S, \partial IB'^{-}_S \setminus \text{int}A'_S)$.

$H_2(I{B'}_S^-, \partial I{B'}_S^- \setminus \text{int}A'_S; \mathbb{Z}) =< c_1, \ldots, c_m >$ represented by a set of vertical annuli, whose Thurston norm vanishes.

$H_2(G'_S, \partial G'_S \setminus A'_S; \mathbb{Z}) =< b_1, \ldots, b_n >$

$\Rightarrow H_2(M_S, \partial M_S; \mathbb{Z}) =$
$< \phi(b_1), \ldots, \phi(b_n), \psi(c_1), \ldots, \psi(c_m) >$.

$\Rightarrow TN(M_S, \partial M_S) \leq TN(\alpha) \leq TN(\beta)$

because: $\|\phi(b_i)\| \leq \|b_i\|$ for $i = 1, ..., n$,

and $0 \leq \|\phi(c_j)\| \leq \|c_j\| = 0$, for $j = 1, ..., m$.

$\Rightarrow$
$TN(M_S, \partial M_S) \leq TN(G'_S, \partial G'_S \setminus \text{int}A'_S)$. $\square$

Proof of the finiteness of the JSJ-pieces

Let M be closed and orientable. We may assume that M is irreducible (even hyperbolic).

$h(M) =$ maximal number of pairwise disjoint, non-parallel, closed, connected, incompressible surfaces embedded in M.

Lemma 2. *Let M and N be two closed, irreducible and orientable 3-manifolds. If M dominates N, then $h(M) \geq h(N)$.*

If M dominates $N \Rightarrow$ number of JSJ-pieces of $N \leq h(M) + 1$.

$\mathcal{M} = \{M_S, \tilde{M}_S\}$, where S runs over all imcompressible surfaces in M and $\tilde{M}_S$ runs over all double covering of M_S.

A closed 3-manifold has only finitely many double coverings, so finiteness of patterned guts and of their Thurston's norm $\Rightarrow$.

Claim 1. *Let M be a closed, irreducible 3-manifold. Then*

(1) Each component of the patterned guts (G_X, A_X) of manifolds $X \in \mathcal{M}$ belongs to a finite set $\tilde{\mathcal{G}}(M)$ of connected compact ∂-patterned 3-manifolds.

(2) $Sup\{TN(X, \partial X) | X \in \mathcal{M}\} \leq L(M)$ for some constant $L(M) > 0$ depending only on M. $\square$

Claim 2. *Fix an integer* $L > 0$*. Then any orientable Seifert manifold* N *with* $\mathbb{H}^2 \times I$ *geometry and which is dominated by a compact orientable 3-manifold* P *with* $TN(P, \partial P) \leq L$ *belongs to a finite set* $\mathcal{S}(L)$ *of compact Seifert 3-manifolds.*

Proof of Claim 2

Assume that the basis of N is orientable.

A homology class y of $H_2(N, \partial N; \mathbb{Z})$ can be represented by an essential surface isotopic to either a vertical torus or annulus, or to a horizontal surface.

Moreover N always admits horizontal surfaces.

Let $\mathcal{O}$ be the orbifold base of N and h be a regular fiber of N. Suppose also that $\mathcal{O}$, h and N are compatibly oriented.

Let F be a horizontal surface of N and $p : F \to \mathcal{O}$ the branched covering, induced by the projection $N \to \mathcal{O}$.

$\chi(F) = |d| \times \chi(\mathcal{O})$, where
$d = \deg(p) = [F] \cdot [h] \neq 0$.

Choose F with minimal genus among all horizontal surfaces in N.

If $\chi(F) \geq 0$, N is homeomorphic to a solid torus, an S^1-bundle over the annulus or a twisted S^1-bundle over a Möbius band.

We can assume $\chi(F) < 0$.

Let $\| \|_N$ (resp. $\| \|_P$) be the Thurston norm on $H_2(N, \partial N; \mathbb{Z})$ (resp. $H_2(P, \partial P; \mathbb{Z})$).

$H_2(N, \partial N; \mathbb{Z})$ is the integer lattice of $H_2(N, \partial N; \mathbb{R})$.

Let $V = \{y \in H_2(N, \partial N; \mathbb{Z}); \|y\|_N = 0\}$ the sublattice of $H_2(N, \partial N; \mathbb{Z})$ generated by the vertical tori and annuli.

Lemma 3. $H_2(N, \partial N; \mathbb{Z}) = \langle [F] \rangle \oplus V$.

□

By hypothesis, there is a compact, orientable 3-manifold P with $TN(P, \partial P) \leq L$ and a non-zero degree map $f : P \to N$.

Let $\alpha = \{z_1, ..., z_m\}$ be a basis of $H_2(P, \partial P; \mathbb{Z})$ realizing $TN(P)$: $\max\{\|z_i\|_P; i = 1, ..., m\} \leq L$.

For $i = 1, ..., m$, let S_i be a properly embedded surface in P representing z_i with $-\chi(S_i) = \|z_i\|_P$.

For $i = 1, ..., m$, set $y_i = [f(S_i)] = \ell_i[F] + v_i \in H_2(N, \partial N; \mathbb{Z})$, where $v_i \in V$.

Triangle inequality and $\|v_i\|_N = 0 \Rightarrow$

$$|\ell_i| \|[F]\|_N = \|\ell_i[F]\|_N = \|y_i - v_i\|_N \leq \|y_i\|_N + \|v_i\|_N = \|y_i\|_N$$

$\|y_i\|_N$ can be calculated using singular surfaces $\Rightarrow$

$\|y_i\|_N \leq -\chi(S_i) = \|z_i\|_P \leq L$.

Combining both inequalities $\Rightarrow$

$\ell_i \|[F]\|_N \leq L$ for $i = 1, ..., m$.

$f : P \to N$ has non-zero degree $\Rightarrow$
$f_*(H_2(P, \partial P; \mathbb{Z})$ has finite index in
$H_2(N, \partial N; \mathbb{Z}) \Rightarrow \not\subset V$.

$\Rightarrow$ some index $i \in \{1, ..., m\}$ verifies $|\ell_i| > 0$

$\Rightarrow \|[F]\|_N \leq L$.

$\Rightarrow$ F can have only finitely many topological types, up to homeomorphism.

N is a surface bundle over S^1 with fiber F and periodic monodromy $g : F \to F$.

Up to conjugacy, a compact surface admits only finitely many periodic homeomorphisms

$\Rightarrow$ N can have only finitely many possible topological types since F has only finitely many topological types. $\square$

Proof of the local finiteness theorem

Let $f : M \to N$ be a map of non-zero degree. After a homotopy of f we may assume that $f^{-1}(\mathcal{T})$ is a non-empty collection of disjoint non-parallel closed incompressible surfaces in M.

Let $\mathcal{M}_f \subset \mathcal{M}$ be the union of all components of $M \setminus f^{-1}(\mathcal{T})$ and of all their double coverings.

Assume that N is not a Seifert manifold supporting the geometries of $\mathbb{S}^3$, $\widetilde{PSL_2(\mathbb{R})}, Nil$.

Soma's results ⇒ the finiteness for hyperbolic JSJ-pieces holds.

A Seifert fibered JSJ-piece $N_i \subset N$ is dominated by at least one component M_i of $M \setminus f^{-1}(\mathcal{T})$.

Claims 1 and 2 ⇒ the finiteness of such JSJ-pieces N_i. □

Theorem 2 *Any closed orientable 3-manifold dominates only finitely many geometrizable integral homology spheres.*

Fix a closed orientable 3-manifold Mthat we can choose irreducible.

Standard arguments reduce the proof to show the finiteness of the set

$\mathcal{D}(M) =$ homeomorphism classes of irreducible, geometrizable, integral homology 3-spheres N which are dominated by M

Lemma 4. *Only finitely many Seifert fibered integral homology 3-spheres belong to* $\mathcal{D}(M)$.

Proof. A closed Seifert fibered integral homology 3-sphere must support the geometry of either S^3 or $\widetilde{PSL_2(\mathbb{R})}$.

There are only two Seifert homology spheres supporting the geometry S^3.

Seifert volume gives a uniform upper bound on the product of the orders of the singular fibers for the $\widetilde{PSL_2(\mathbb{R})}$ homology spheres in $\mathcal{D}(M)$. $\Rightarrow$ finiteness. $\square$

Denote by $\Gamma(N)$ the tree dual to the JSJ-decomposition of an irreducible homology sphere N.

Fix a tree Γ.

$\mathcal{D}(M,\Gamma) \subset \mathcal{D}(M)$ = geometrizable integral homology 3-spheres N such that:

(1) N is dominated by M.

(2) The JSJ-graph $\Gamma(N) = \Gamma$.

(3) Each vertex manifold has a fixed topological type.

The number of edges of $\Gamma(N)$ is $\leq h(M)$, the Haken number of M.

Proposition 2. *The set $\mathcal{D}(M, \Gamma)$ is finite.*

Let $N \in \mathcal{D}(M, \Gamma)$.

Let $e \in \Gamma$ be an edge corresponding to an incompressible torus T_e in the JSJ-splitting.

T_e splits N into two integral homology solid tori V and W

On each torus ∂V and ∂W fix a homology basis $\{\mu_V, \lambda_V\}$ and $\{\mu_W, \lambda_W\}$ such that:

(1) $\mu_V \subset \partial V$ (resp. $\mu_W \subset \partial W$) bounds a properly embedded, essential surface F_V in V (resp. F_W in W).

(2) The intersection numbers $\mu_V \cdot \lambda_V = \mu_W \cdot \lambda_W = 1$

(3) λ_V generates $H_1(V; \mathbb{Z})$ and λ_W generates $H_1(W; \mathbb{Z})$

Lemma 5. *The gluing map $\phi : \partial V \to \partial W$ induces a map $\phi_\star$ on the first homology group, which satisfies the following equations, where $\varepsilon = \pm 1$, $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$:*

(1) $\phi_\star(\mu_V) = p\mu_W + \epsilon\lambda_W,$

(2) $\phi_\star(\lambda_V) = \varepsilon(pq + 1)\mu_W + q\lambda_W.$

Pinching the surface F_V onto a disk D^2, gives a proper degree-one map
$p_V : V \to S^1 \times D^2$ such that :

$p_V^{-1}(\{x\} \times \partial D^2) = \mu_V$ for some point $x \in S^1$.

$\Rightarrow$ a degree-one map $f_V : N \to W(p/\varepsilon)$

$\Rightarrow W(p/\varepsilon) \in \mathcal{D}(M)$.

The homology sphere $W(p/\varepsilon)$ is obtained by Dehn filling W with a solid torus via the gluing map ϕ:

$\phi_\star(\partial D^2) = p\mu_W + \epsilon\lambda_W.$

Lemma 6. *The integer p takes only finitely many values.*

Proof. Let $Y \subset W$ be the JSJ-piece of N adjacent to T_e.

Y has a fixed topological type and is either hyperbolic or Seifert fibered.

Denote by $Y(p/\varepsilon)$ the Dehn filling of Y along $T_e \subset \partial Y$

Standard Dehn filling arguments $\Rightarrow$ for almost all values of p, $Y(p/\varepsilon)$ is hyperbolic or the Seifert fibration of Y extends to $Y(p/\varepsilon)$.

$\Rightarrow Y(p/\varepsilon)$ is a JSJ-piece of the integral homology sphere $W(p/\varepsilon) \in \mathcal{D}(M)$.

Finiteness of JSJ -pieces in $\mathcal{D}(M) \Rightarrow$

Claim 3. *The closed manifold $Y(p/\varepsilon)$ belongs to only finitely many topological types.*

Now Proposition 6 follows from:

Claim 4. *There are only finitely many values of p such that the manifold $Y(p/\varepsilon)$ has a fixed topological type.*

Proof. If Y is hyperbolic the Claim follows from Thurston's hyperbolic Dehn filling theorem .

Otherwise Y is Seifert fibered and for almost all p the Seifert fibration extends to $Y(p/\varepsilon)$.

The order $a \geq 1$ of the possibly exceptional fiber corresponding to the core of the Dehn filling is given by:

$$a = \phi_\star(\partial D^2) \cdot h = p(\mu_W \cdot h) + \varepsilon(\lambda_W \cdot h).$$

This order a is determined by the topological types of $Y(p/\varepsilon)$ and Y $\Rightarrow$

finitely many values for p, because $\mu_W \cdot h \neq 0$. $\square$