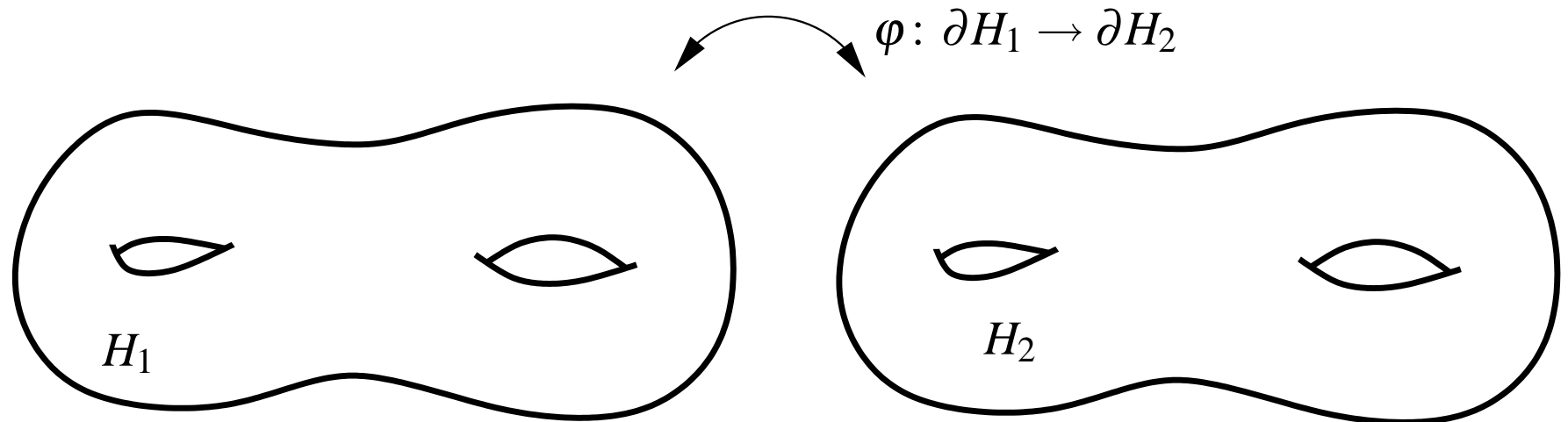


The geometry of hyperbolic 3-manifolds via Heegaard splittings

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The Goal

In this part we will tie together:

- Heegaard splittings
- Pants decompositions of surfaces
- Volumes of hyperbolic 3-manifolds

Or attempt to answer the question:



is there life after Perelman?

Certainly *Yes*

Existence and uniqueness of hyperbolic structures on manifolds give little information about their structure.

Motivating Question:

Given a **combinatorial description** of a 3-manifold that admits a hyperbolic structure what can be said about the **geometry** of that structure?

Effective Rigidity

For M a closed or finite volume hyperbolic manifold, **Mostow rigidity** guarantees uniqueness of the hyperbolic structure.

More **effective** information could include:

- The volume of M ,
- The injectivity radius of M ,
- The Cheeger constant for M , or
- A bi-Lipschitz model for M .

A Sample Case

The 3-manifold M *fibers over the circle with fiber S and monodromy $\psi \in \text{Diff}^+(S)$ if*

$$M \cong M_\psi = S \times [0, 1] / (x, 0) \sim (\psi(x), 1).$$

Theorem. (Thurston) *The 3-manifold M_ψ is hyperbolic if and only if ψ is pseudo-Anosov.*

Definition. ψ is *pseudo-Anosov* if no power of it preserves the isotopy class of any essential simple closed curve on S .

A Sample Theorem

Theorem. (B-'01) *There is a K depending only on S so that*

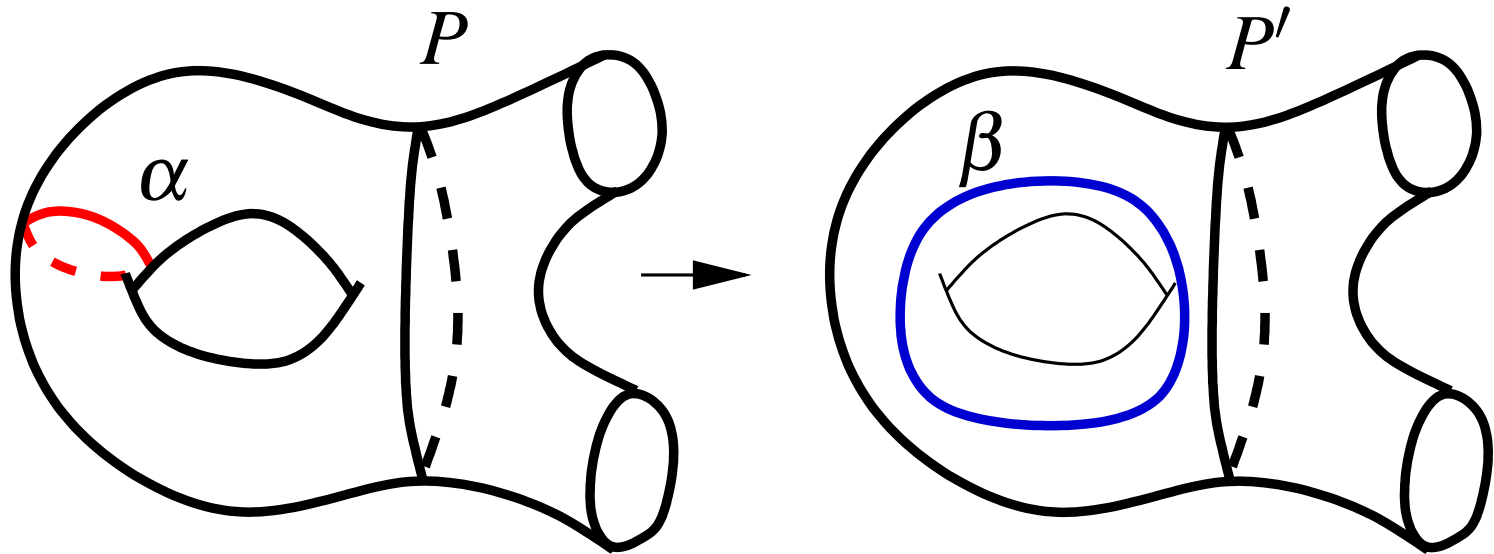
$$\text{vol}(M_\psi) \in [K^{-1}, K] \|\psi\|_P.$$

Here, the quantity $\|\psi\|_P$ measures the *translation distance* of ψ on the *pants graph*.

The theorem presages the extra control of the model theorem.

The Pants Graph

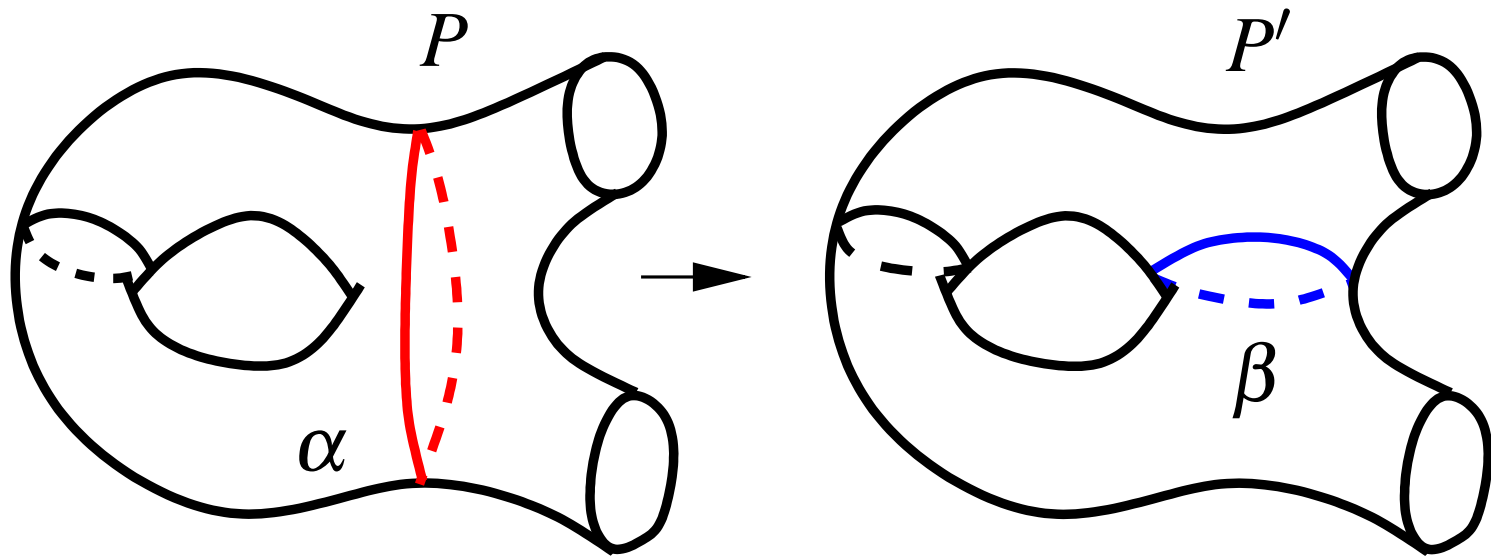
The pants graph $P(S)$ has a vertex for each pants decomposition of S and edges joining vertices whose pants decompositions differ by an **elementary move**:



a type-1 move

The Pants Graph

The pants graph $P(S)$ has a vertex for each pants decomposition of S and edges joining vertices whose pants decompositions differ by an **elementary move**:



a type-2 move

Translation Length

Giving each edge length 1, we obtain a metric space and a notion of *pants distance*

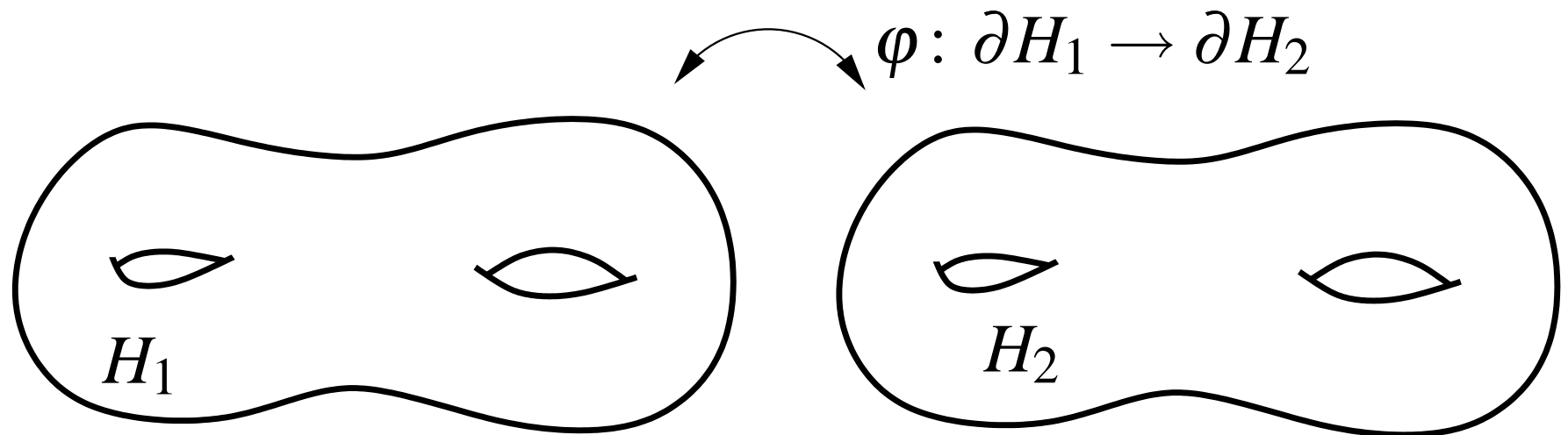
$$d_P: P^0(S) \times P^0(S) \rightarrow \mathbb{Z}$$

between pants decompositions of S .

- The mapping class ψ acts by isometries of $P(S)$.
- The translation length $\|\psi\|_P$ of ψ is defined to be the min of $d_P(P, \psi(P))$ over all P .
- Like **Weil-Petersson** translation distance $\|\psi\|_{\text{WP}}$.

Heegaard Splittings

Heegaard splittings represent another instance where surfaces give a combinatorial description of 3-manifolds.



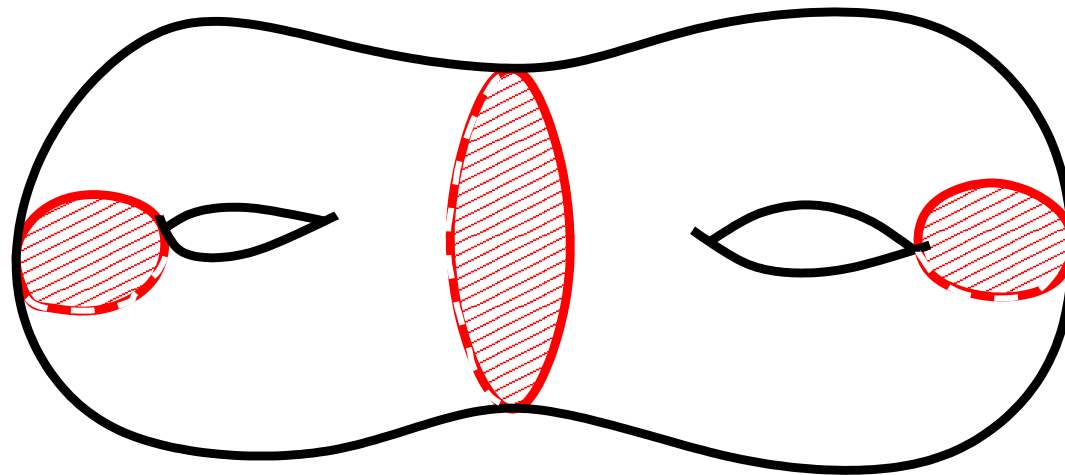
Can we estimate the volume of a hyperbolic Heegaard splitting via its combinatorics?

Meridians

One question:

How do pants decompositions arise?

A *meridian* on H is an essential simple closed curve on ∂H that bounds a disk in H .



(Many meridional pants decompositions of ∂H .)

Handlebody Sets

Consider the set $\mathcal{M}(H) \subset P(\partial H)$ consisting of pants decompositions consisting of meridians.

- Consider a hyperbolic Heegaard splitting

$$M = H_1 \cup H_2$$

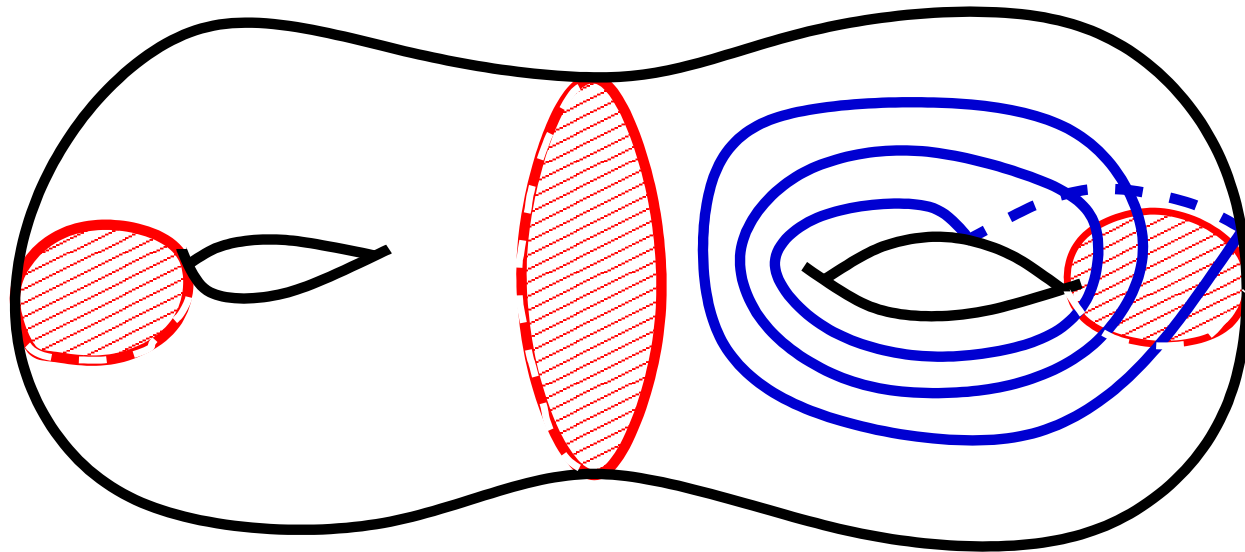
- One is tempted to guess that

$$d_P(\mathcal{M}(H_1), \mathcal{M}(H_2))$$

may bear some relation to volume...

Not Quite...

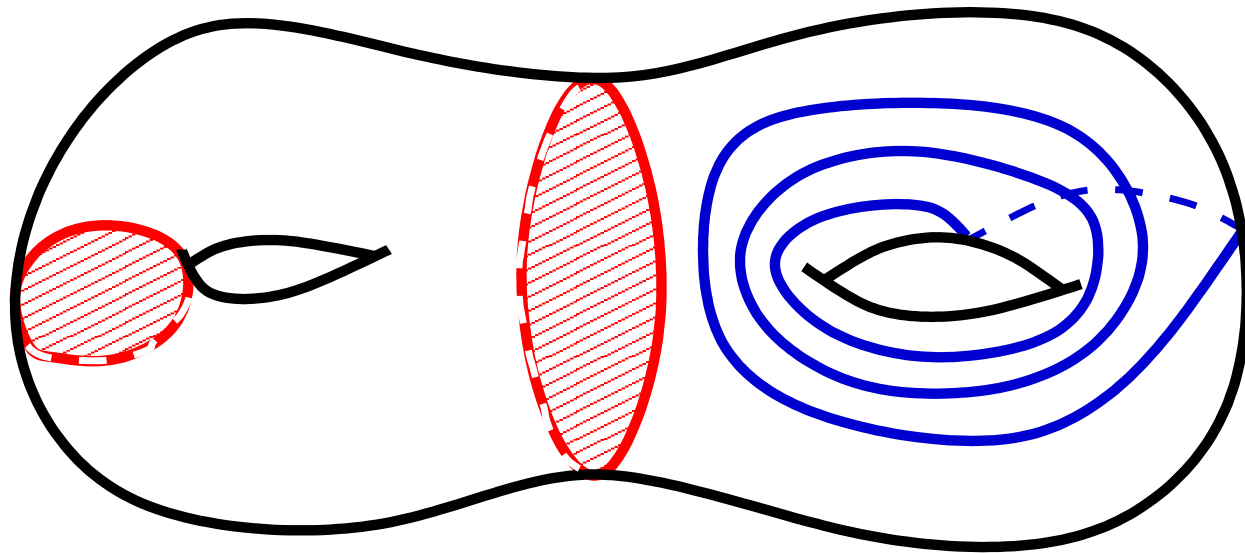
Dehn surgery about the core of a handle can increase distance to $\mathcal{M}(H)$ arbitrarily, but volume remains bounded.



Handlebody Sets II

As such, we modify our handlebody set:

Definition. Let $\mathcal{P}(H)$ denote the set of pants decompositions of ∂H so that each $P \in \mathcal{P}(H)$ contains a set of meridians whose meridian disks *decompose* $\text{int}(H)$ into solid tori.



Heegaard Volumes

Then we have the following:

Theorem. (B-Souto) *Let $M = H_1 \cup_{\varphi} H_2$ be a **strongly irreducible** Heegaard splitting of $M = \mathbb{H}^3 / \Gamma$. Then there is a K depending only on $\partial H_i = S$ so that*

$$\text{vol}(M) \in [K^{-1}, K] d_P(\mathcal{P}(H_1), \mathcal{P}(H_2)).$$

The Heegaard splitting $M = H_1 \cup H_2$ is **strongly irreducible** if every meridian of H_1 intersects every meridian of H_2 (or **Heegaard distance** > 1).

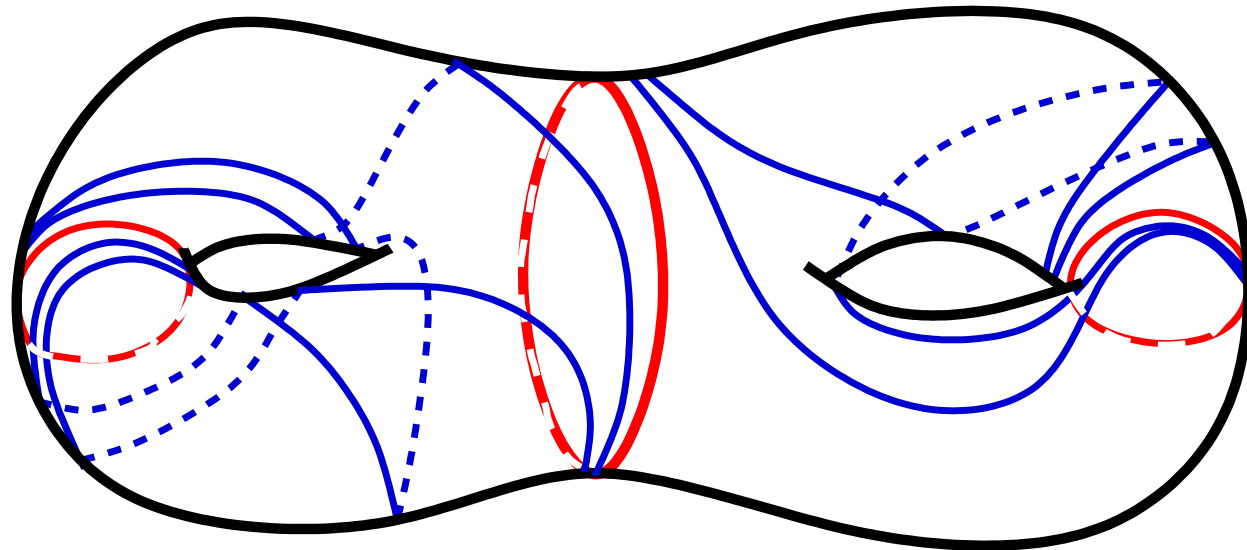
The Upper Bound

Elements of the Proof.

- To bound of volume from above in terms of $d_P(\mathcal{P}(H_1), \mathcal{P}(H_2))$ is a fairly standard argument:
- A *spun ideal triangulation* of M adapted to the pants decompositions in a shortest path in $P(S)$ joining $\mathcal{P}(H_1)$ to $\mathcal{P}(H_2)$ has a bounded number of tetrahedra for each elementary move.

Ideal Triangulations

The triangulation interpolates between ideal surface triangulations.



Each ideal hyperbolic tetrahedron has volume at most that of the regular ideal tetrahedron, giving the upper bound.

The Lower Bound

- The lower bound is somewhat more delicate.
- Strong irreducibility provides a **minimal Heegaard surface X** (Pitts-Rubinstein).
- Then $M \setminus X$ is a union of two hyperbolic handlebodies with boundary a surface of negative curvature.

The Handlebody Case

This essentially reduces the theorem to estimating the volume of the convex core of a **hyperbolic handlebody** $N = \mathbb{H}^3 / \Gamma \cong \text{int}(H)$.

Theorem. (Bers) *There is an L depending only on ∂H so that $\partial CC(N)$ admits a pants decomposition of length less than L .*

If $\partial CC(N) = Y$, then we let P_Y be this **bounded length** pants decomposition.

Then we have...

Handlebody Volume

Theorem. (B-Souto) *The convex core volume of N satisfies*

$$\text{vol}(CC(N)) \in [K^{-1}, K] d_P(P_Y, \mathcal{P}(H)).$$

Similar statements exist for **compression bodies**.

Bounded Geometry Case

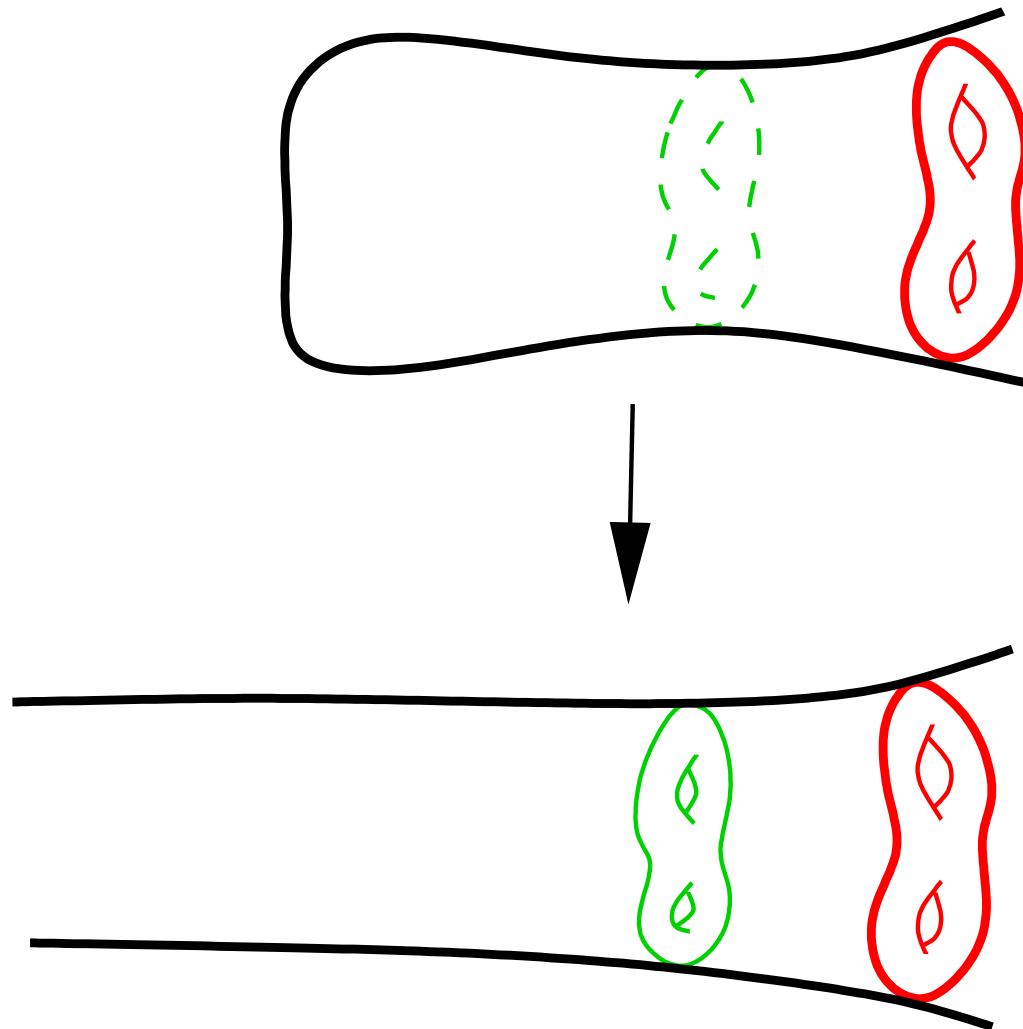
When $\text{inj}(N) > \varepsilon_0$ we have the following:

Claim. *There exist $D, L > 0$ so that if the shortest meridian on Y has length at least L there is an embedded uniformly Lipschitz hyperbolic surface Y' in $CC(N)$ so that*

- Y and Y' cut off a product with volume at least 1, and
- We have $d_P(P_Y, P_{Y'}) \leq D$.
- A **geometric limit** argument...

Geometric Limits

The worse things are the better they are...



Bounds from Below

- Proceeding inductively, we bound the pants distance by the volume:
- As long as the shortest meridian has length at least L , there is another **unit volume product** within **bounded pants distance**.
- Proceed until short meridians appear, **pinch** and continue.
- The process terminates at bounded volume tori.

Unbounded geometry

- A theorem of Souto guarantees that short geodesics in M are **unknotted and unlinked** (they can all be put on disjoint *level surfaces*).
- Drilling reduces to the bounded geometry case.

Theorem (B-Bromberg). *Let M be a geometrically finite cone-manifold. For all $K > 1$ there exists an $\varepsilon > 0$ so that if c is a closed geodesic in M with $\ell_M(c) < \varepsilon$, then there is a K -bi-Lipschitz diffeo.*

$$\phi : M \setminus U_c \rightarrow M_0 \setminus \mathbb{P}_c.$$

Other Questions

- Can one predict short curves from the combinatorics of a Heegaard splitting?
- Is there a natural **model manifold** arising from these combinatorics? (cf. B-Canary-Minsky)
- Can combinatorics yield hyperbolic structure?

Theorem (Namazi). *Each M with a Heegaard splitting $M = H_1 \cup H_2$ of sufficiently large \mathcal{C} -distance and **R -bounded combinatorics** is hyperbolic.*

Thick Models

In recent work we have obtained the following:

Theorem (B-Souto). *Let M be a hyperbolic structure on a handlebody with $\text{inj}(M) > \varepsilon$ and conformal boundary X . Then the pants decomposition P_X determines a uniform bi-Lipschitz model for M realized as a gluing of*

- *thick products $S_i \times I$, with $g(S_i) \leq g(X)$,*
- *bounded diameter two-handles and handlebodies.*