# Finite covers of 3-manifolds, II

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## Subgroup growth

$s_n(\Gamma)$ = number of subgroups of $\Gamma$ with index at most $n$
= number of covering spaces of $M$ with degree at most $n$.

Theorem 2.1: [L]

$$s_n(\Gamma) \geq 2^{n/(\sqrt{\log n}\log\log n)}$$

for infinitely many $n$.

Compare: $s_n(\text{non-abelian free group}) \simeq 2^{n\log n}$.

This is the fastest subgroup growth among all fin. gen. groups.

Recall:

Theorem 1.2: [Lubotzky] For any prime $p$,

$$\sup\{d_p(\Gamma_1) : \Gamma_1 \text{ is a f.i. subgroup of } \Gamma\} = \infty.$$

Define the derived $p$-series of a group $G$ to be:

$$G_1 = G, \qquad G_{i+1} = \Phi_p(G_i) = [G_i, G_i](G_i)^p$$

Note that $G_i/G_{i+1} = H_1(G_i; \mathbb{F}_p)$.

Theorem 2.2: [L] Let $\Gamma_1$ be a finite index subgroup of $\Gamma$ s.t. $d_2(\Gamma_1) > 3$. Let $\Gamma_i$ be the derived 2-series of $\Gamma_1$. Then

$$d_2(\Gamma_i) = \Omega\left(\frac{[\Gamma : \Gamma_i]}{\sqrt{\log[\Gamma : \Gamma_i]\log\log[\Gamma : \Gamma_i]}}\right).$$

This is nearly the maximal possible growth rate for $d_2(\Gamma_i)$, since by Reidermeister-Schreier

$$d_2(\Gamma_i) \leq O([\Gamma : \Gamma_i]).$$

Thm 2.2 $\Rightarrow$ Thm 2.1:

The number of subgroups of $\Gamma_i$ with index 1 or 2 is $2^{d_2(\Gamma_i)}$. Setting $n = [\Gamma : \Gamma_i]$,

$$s_{2n}(\Gamma) \geq 2^{d_2(\Gamma_i)} > 2^{n/(\sqrt{\log n} \log \log n)}$$

## Facts about $p$-groups

Let $G$ = any finite $p$-group.

Fact 2.3: The derived $p$-series of $G$ terminates.

Corollary 2.4: Let $\tilde{M} \to M$ be a regular cover with degree a power of $p$. This factorises as

$$\tilde{M} = M_n \to M_{n-1} \to \ldots \to M_0 = M$$

where successive covers are regular with covering group $\mathbb{Z}/p\mathbb{Z}$.

Proof: Let $\tilde{\Gamma} = \pi_1(\tilde{M})$. Let $G = \Gamma/\tilde{\Gamma}$.

Let $\phi\colon \Gamma \to G$ be the quotient hm.

The derived $p$-series for $G$:

$$G = G_1 \supset G_2 \supset \ldots \supset G_n = 1$$

has $G_i/G_{i+1} = (\mathbb{Z}/p\mathbb{Z})^{d_p(G_i)}$.

Refine this so that $G_i/G_{i+1} = \mathbb{Z}/p\mathbb{Z}$.

Let $\Gamma_i = \phi^{-1}(G_i)$ and let $M_i$ be the corresponding covering space.

Fact 2.5: $d(G) = d_p(G)$.

Here, $d(\ )$ = the rank of a group.

## Applications

Prop 2.6: Suppose that $d_p(\Gamma) = 0$. Then the only normal subgroup with index a power of $p$ is $\Gamma$.

Proof: $\Gamma$ has no regular covers with covering group $\mathbb{Z}/p\mathbb{Z}$. Now use 2.4.

Prop 2.7: Suppose that $d_p(\Gamma) = 1$. Then, for any normal subgroup $\Gamma_1$ with index a power of $p$, $\Gamma/\Gamma_1$ is cyclic.

Proof: $\Gamma/\Gamma_1$ is a $p$-group with

$$d(\Gamma/\Gamma_1) = d_p(\Gamma/\Gamma_1) \leq d_p(\Gamma) = 1.$$

Prop 2.8: Let $G$ be a fin gen group, and let $G_1$ be a normal subgroup with index a power of $p$. Then

$$d_p(G) - d_p(G/G_1) \leq d_p(G_1)$$
$$\leq (d_p(G) - 1)[G : G_1] + 1.$$

c.f. Reidermeister-Schreier:

$$d(G_1) \leq (d(G) - 1)[G : G_1] + 1.$$

Proof:

$$1 \to G_1 \to G \to G/G_1 \to 1$$

is exact and hence so is

$$1 \to G_1/\Phi_p(G_1) \to G/\Phi_p(G_1) \to G/G_1 \to 1$$

When we have an exact sequence of groups:

$$d(G/\Phi_p(G_1)) \leq d(G_1/\Phi_p(G_1)) + d(G/G_1)$$

These are all finite $p$-groups and so:

$$d_p(G/\Phi_p(G_1)) \leq d_p(G_1/\Phi_p(G_1)) + d_p(G/G_1)$$

$$= d_p(G) \qquad\qquad = d_p(G_1)$$

For the other inequality:

$G_1/\Phi_p(G_1)$ is a normal subgroup of $G/\Phi_p(G_1)$ with index $[G : G_1]$. Hence, by Reidermeister-Schreier:

$$d(G_1/\Phi_p(G_1)) \leq (d(G/\Phi_p(G_1)) - 1)[G : G_1] + 1.$$

$$= d_p(G_1) \qquad\qquad = d_p(G)$$

# Homology growth for 3-manifolds

Let $\Gamma_1$ = f.i. subgroup of $\Gamma$.
Let $\Gamma_2 = \Phi_p(\Gamma_1) = [\Gamma_1, \Gamma_1](\Gamma_1)^p$.

Theorem 2.9: [Shalen-Wagreich]

$$d_p(\Gamma_2) \geq \binom{d_p(\Gamma_1)}{2}.$$

e.g. 3-torus

But if $d_p(\Gamma_1) > 3$, then $d_p(\Gamma_2) > d_p(\Gamma_1)$.

$\rightarrow$ quite fast homology growth

Theorem 2.9 is a special case of a more general result:

Theorem 2.10: [Shalen-Wagreich] Let $G_1$ be a group s.t. $d_p(G_1)$ and $b_2(G_1;\mathbb{F}_p)$ are finite. Let $G_2 = \Phi_p(G_1)$. Then

$$d_p(G_2) \geq \binom{d_p(G_1)}{2} + d_p(G_1) - b_2(G_1;\mathbb{F}_p).$$

Note: $d_p(\Gamma_1) = b_2(\Gamma_1;\mathbb{F}_p)$

We'll prove a slightly weaker result:

Theorem 2.11: Let $G_1 = \langle X|R\rangle$ be a finitely presented group. Then

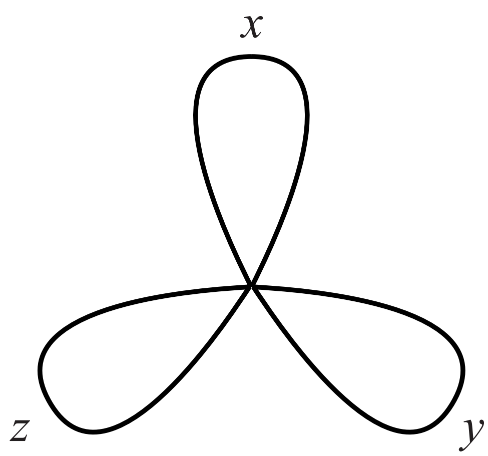
$$d_p(G_2) \geq \binom{d_p(G_1)}{2} + d_p(G_1) - |R|.$$

## Proof

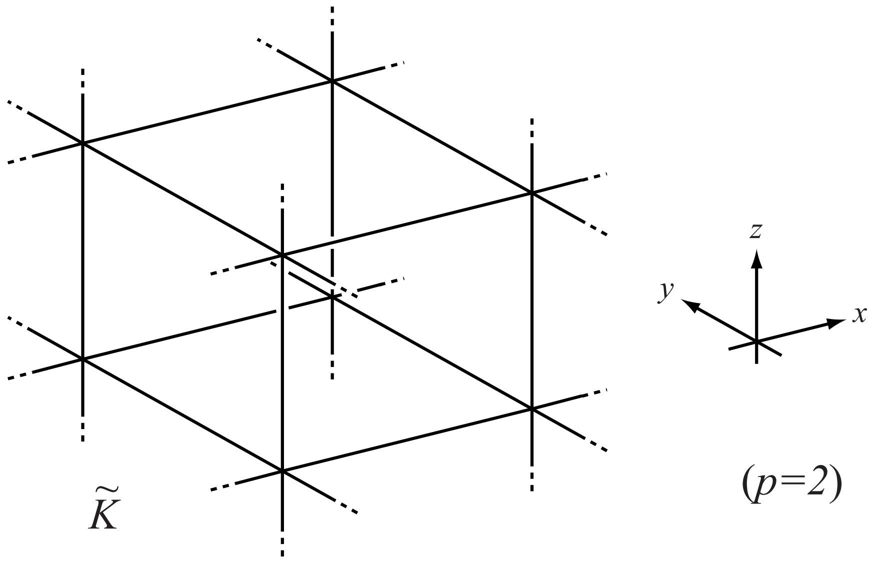
Let $d = d_p(G_1)$.

We may assume that the first $d$ elements of $X$ form a basis for $H_1(G_1; \mathbb{F}_p)$, and that the rest are trivial in $H_1(G_1; \mathbb{F}_p)$.

Let $K$ = 2-complex arising from presentation of $G_1$ e.g:




Let $\tilde{K}$ = covering space corr. to $G_2$.




$$d_p(G_2) = \dim\ H^1(\tilde{K}; \mathbb{F}_p)$$

$$H^1(\tilde{K}; \mathbb{F}_p) = \frac{\{\text{1-cocycles}\}}{\{\text{1-coboundaries}\}}$$

A 1-cochain is an assignment of an integer mod $p$ to each oriented edge.

A 1-cocycle is a 1-cochain that has zero evaluation around each 2-cell.

A 1-coboundary is a 1-cochain that has zero evaluation around each closed loop.

Dimension of space of cochains on $\tilde{K}$? $|X|p^d$

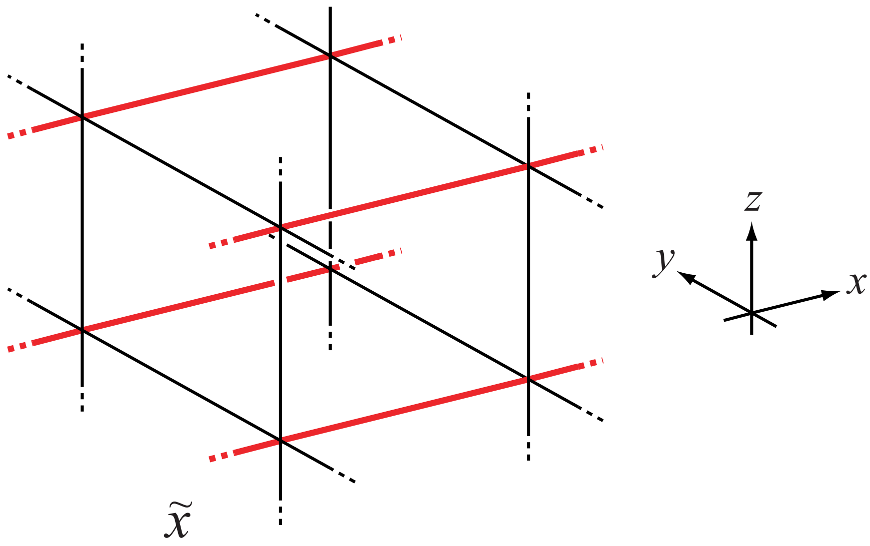
How many conditions are required to specify a cocycle? $|R|p^d$.

Since $|R| = |X|$ in general for 3-manifolds, this gives no information.

The trick: Use special cochains.

Let $x$ be a cocycle on $K$ representing a non-trivial element of $H^1(K; \mathbb{F}_p)$.

Its inverse image in $\tilde{K}$ is a cocycle $\tilde{x}$.

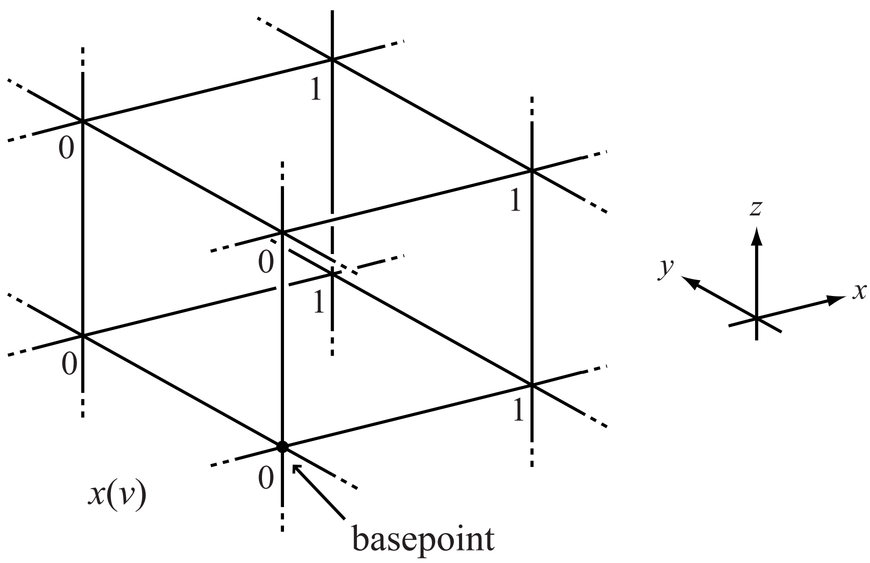


Each vertex $v$ in $\tilde{K}$ has a well-defined value $x(v)$:

Pick a path $p$ from the basepoint to $v$. Let $x(v) = \tilde{x}(p)$.

Independent of path $p$ because

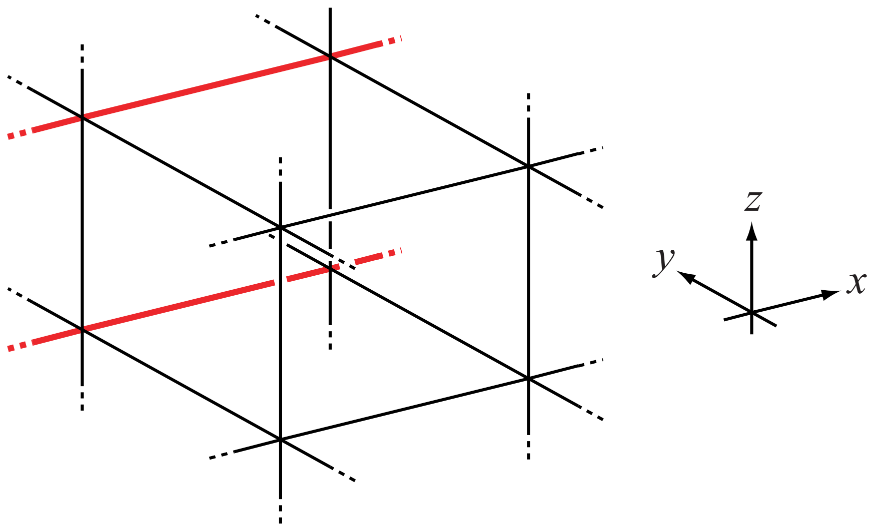
$$\tilde{x}(\text{closed loop in } \tilde{K}) = 0.$$



Given two cocycles $x$ and $y$ on $K$, define the cochain $x \wedge y$ on $\tilde{K}$:

$$(x \wedge y)(e) = \tilde{x}(e) \, . \, y(i(e)),$$

where $e$ is an oriented 1-cell of $\tilde{K}$ and $i(e)$ is the initial vertex of $e$.




Key claim: Let $\tau$ be a covering transformation of $\tilde{K}$. Let $L$ be the boundary of a 2-cell. Then

$$(x \wedge y)(L) = (x \wedge y)(\tau L).$$

Proof: $\tau$ is effected by conjugating by some $g \in G_1$.

$$(x \wedge y)(\tau e) - (x \wedge y)(e) \\ = \tilde{x}(\tau e)y(i(\tau e)) - \tilde{x}(e)y(i(e)) = \tilde{x}(e)y(g)$$

So,

$$(x \wedge y)(\tau L) - (x \wedge y)(L) = \tilde{x}(L)y(g) = 0$$

because $\tilde{x}(\text{any closed loop}) = 0$.

So, the space of cocycles has dimension $\geq$

$$\begin{aligned} &\binom{d}{2} && : \{x_i \wedge x_j\}, i \neq j \\ &+d && : \{x_i \wedge x_i\} \\ &-|R| && : \text{can in fact be replaced by } b_2 \end{aligned}$$

In fact, none of these cocycles are coboundaries (except the zero cocycle) when $p = 2$.

□ Thm 2.11

Thm 2.12: [L] Let $G_1$ be a group where $d_2(G_1)$ and $b_2(G_1;\mathbb{F}_2)$ are finite. Consider $G_2 \triangleleft G_1$ s.t. $G_1/G_2 \cong (\mathbb{Z}/2\mathbb{Z})^n$. Then, for any integer $\ell$ between 0 and $n$,

$$d_2(G_2) \geq (d_2(G_1) - 1) \sum_{r=0}^{\ell} \binom{n}{r} + 1 - \binom{n}{\ell+1} - b_2(G_1;\mathbb{F}_2) \sum_{r=0}^{\ell-1} \binom{n}{r}.$$

Important terms are in red.

Setting $n = d_2(G_1)$ and $\ell = 1$ gives [Shalen-Wagreich]

Optimal choice: Set $n = d_2(\Gamma_1)$ and $\ell = \lfloor n/2 \rfloor$:

$$d_2(\Gamma_2) \gtrsim n \binom{n}{n/2} \simeq \sqrt{n}\, 2^n.$$

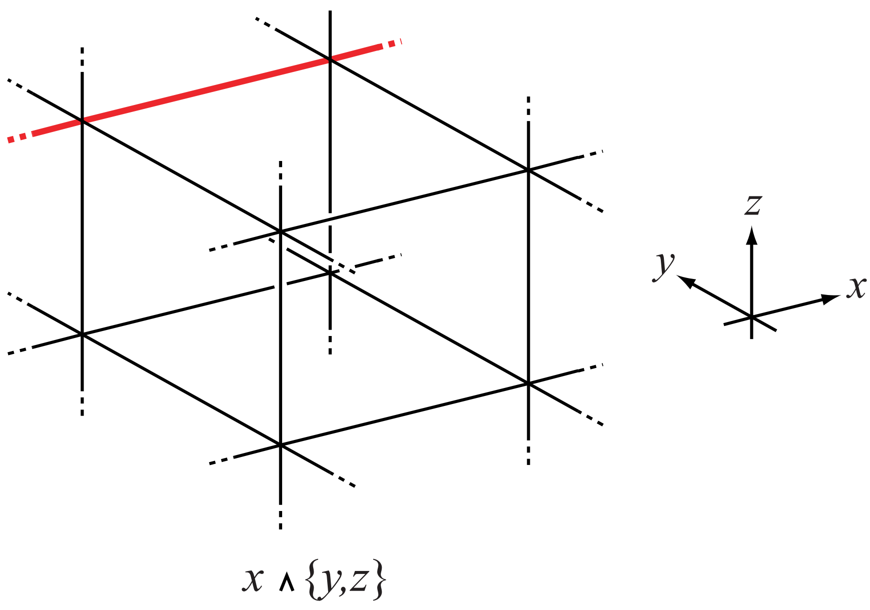
This $\Rightarrow$ 2.2 and 2.1.

## Idea of proof

Use more cocycles.

For any $A \subset \{1, \ldots, d\}$ with size $\leq \ell$, define a cocycle $x \wedge A$:

$$(x \wedge A)(e) = \tilde{x}(e) \prod_{y \in A} y(i(e))$$



Analogue of 'Key Claim': Let

$$N = \sum_{r=0}^{\ell-1} \binom{n}{r}.$$

There exist covering transformations $\tau_1, \ldots, \tau_N$ s.t. if $(x \wedge A)(\tau_i L) = 0$ for $i = 1, \ldots, N$, and some 2-cell $L$, then $(x \wedge A)(\tau L) = 0$ for every covering transformation $\tau$.

□ Thm 2.12

## Fast homology growth and largeness

Thm 2.2 ⇒ hyperbolic 3-manifolds have fast homology growth.

Really fast homology growth ⇒ largeness:

A sequence of f.i. subgroups $G_i$ of $G$ has linear growth of mod $p$ homology if

$$\inf_i d_p(G_i)/[G : G_i] > 0.$$

Thm 2.13: [L] Let $G$ be a fin. pres. group. Suppose that the derived $p$-series of $G$ has linear growth of mod $p$ homology. Then $G$ is large.

The proof uses explicit constructions of cocycles (cf proof of Thm 2.11) and some ideas from the theory of error-correcting codes!

Let $G$ = a fin. pres. group.

Let $\phi: G \to \mathbb{Z}$ be a surj. hm.

Let $G_i = \phi^{-1}(i\mathbb{Z})$.

Theorem 2.14: [L] Suppose that $d_p(G_i)$ is unbounded. Then $G$ is large.

Proof.

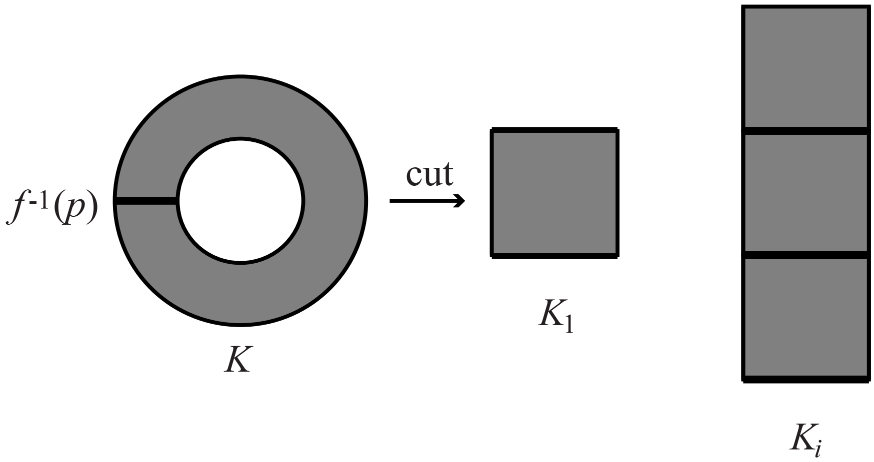
Let $K$ = a finite 2-complex with $\pi_1(K) = G$.

Realise $\phi$ as $f: K \to S^1$. Pick a point $p \in S^1$, where $f^{-1}(p)$ is a 'properly embedded' graph.

Let $K_1 = K$ cut along $f^{-1}(p)$.

Let $K_i = i$ copies of $K_1$ glued together.

Let $\partial K_i$ be the 'top' and 'bottom' of $K_i$.




$d_p(G_i)$ unbounded $\Rightarrow d_p(K_i)$ unbounded.

Each element $z$ of $H^1(K_i;\mathbb{F}_p)$ is represented by a mod $p$ cocycle, which specifies a map $K_i \to L(p)$, where $L(p)$ is the 2-complex with

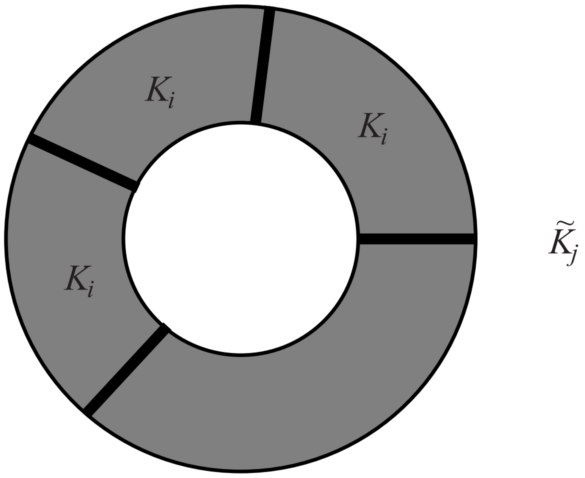
- a single 0-cell
- a single 1-cell
- a 2-cell running $p$ times around the 1-skeleton.

When $d_p(K_i) > d_p(\partial K_i)$, we may find a cocycle on $K_i$ representing a non-trivial element of $H^1(K_i;\mathbb{F}_p)$ supported away from $\partial K_i$.

This gives a fn $f\colon K_i \to L(p)$ s.t.

1. $f_*$ is surjection;
2. $f$ sends $\partial K_i$ to the 0-cell of $L(p)$.

For $j > 3i$, let $\tilde{K}_j$ be the covering space of $K$ corr to $G_j$. Cut $\tilde{K}_j$ into 3 copies of $K_i$ and a copy of $K_{j-3i}$.




We have

$$\tilde{K}_j \to L(p) \vee L(p) \vee L(p)$$

which induces a surjection

$$G_j \to (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z})$$

So, $G$ is large. □ Thm 2.14