Richard Weidmann
Fachbereich Informatik und Mathematik
Johann Wolfgang Goethe-Universität
60054 Frankfurt
Germany

Foldings, the rank problem and Nielsen equivalence

Nielsen equivalence

Let G be a group and $\mathcal{T} = (g_1, \ldots, g_n)$ and $\mathcal{T}' = (g'_1, \ldots, g'_n)$ be two tuples of elements of G. Recall that $\mathcal{T}$ and $\mathcal{T}'$ are called *elementary equivalent* if one of the following holds.

1. There exists some $\sigma \in S_n$ such that $g'_i = g_{\sigma(i)}$ for $1 \leq i \leq n$.
2. $g'_i = g_i^{-1}$ and $g'_j = g_j$ for $j \neq i$.
3. $g'_i = g_i g_j^{\varepsilon}$ for some $i \neq j$ and $\varepsilon \in \{-1, 1\}$. Furtermore $g'_k = g_k$ for $k \neq j$.

We further say that $\mathcal{T}$ and $\mathcal{T}'$ are *Nielsen equivalent* or simply *equivalent* if there exists a sequence of tuples

$$\mathcal{T} = \mathcal{T}_0, \ldots, \mathcal{T}_k = \mathcal{T}'$$

such that $\mathcal{T}_{i-1}$ and $\mathcal{T}_i$ are elementary equivalent for $1 \leq i \leq k$.

Theorem 1 (Nielsen) *Let* $X = (x_1, \ldots, x_n)$*,* $F_n = F(X)$ *the free group of rank* n *and* $\mathcal{T}$ *be a generating tuple of* F_n*. Then* $\mathcal{T}$ *is Nielsen-equivalent to* $(x_1, \ldots, x_n, 1, \ldots, 1)$*.*

This implies that $\mathcal{T} = (g_1, \ldots, g_n)$ and $\mathcal{T}' = (g_1', \ldots, g_n')$ are Nielsen equivalent if and only if there exists a homomorphism $\phi : F_n \to G$ such that

1. $\phi(x_i) = g_i$ for all i for some basis $(x_1, \ldots, x_n)$ of F_n.
2. $\phi(y_i) = g_i'$ for all i for some basis $(y_1, \ldots, y_n)$ of F_n.

Purpose of Nielsen moves: Replace and arbitrary tuple by a more efficient Nielsen-equivalent tuple that is better to understand. Generalizes the Euclidean algorithm.

Theorem 2 (Grushko) *Let $G = H * K$ and $\mathcal{T}$ be a generating tuple of G. Then $\mathcal{T}$ is Nielsen-equivalent to a tuple $(h_1, \ldots, h_l, k_1, \ldots, k_m)$*

Corollary 3 *rank $H * K =$ rank $H +$ rank K.*

Call an action of a group G on a tree T k-acylindrical if the stabilizer of any segment of length k is trivial. A graph of group is k-acylindrical if the action on Bass-Serre tree is acylindrical.

1. Free products are 0-acylindrical.
2. Amalgamated products with malnormal amalgam are 1-acylinrical.
3. JSJ-decompositions are usually 2-acylindrical.

Theorem 4 (W) *Let $\mathbb{A}$ be a k-acylindrical graph of groups without trivial edge group and $G = \pi_1(\mathbb{A})$. Then any generating tuple of G contains only short elements after Nielsen equivalence.*

This implies acylindrical accessibility.

Theorem 5 (Sela,W) *Let $\mathbb{A}$ be a k-acylindrical graph of groups without trivial edge groups and $G = \pi_1(\mathbb{A})$. Then $\mathbb{A}$ has at most $1 + 2k \cdot$ rank G vertices.*

Theorem 6 (Zieschang) *Let* $G = \pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_b \,|\, [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \rangle$. *Then any minimal generating tuple of* G *is Nielsen equivalent to* $(a_1, b_1, \ldots, a_b, b_g)$.

Uniqueness is not always possible, but finiteness very often is.

Theorem 7 (Kapovich,W.) *Let* M *be locally quasi-convex torsion-free hyperbolic group. For any* k *there exist only finitely many Nielsen equivalence classes of generating tuples of* G *of length* k.

With the exception of virtual fibers Kleinian groups are of this type.

Theorem 8 (Kapovich,W.) *Let* M *be a closed hyperbolic 3-manifold and* $G = \pi_1(M)$. *For any* k *there exist only finitely many Nielsen equivalence classes of generating tuples of* G *of length* k.

This is an analogue of the Waldhausen conjecture on the finiteness of isotopy classes of Heegaard splittings, Jaco/Rubinstein and Li.

Li however proves an absolute finiteness result for non-Haken manifolds:

Question 9 *Are there fundamental groups of hyperbolic 3-manifolds such that any generating tuple of length greater than* k *is reducible for some* k*?*

$(g_1, \ldots, g_n)$ is called *reducible* if it is Nielsen equivalent to a tuple of type $(1, h_2, \ldots, h_n)$

Theorem 10 (Souto) *Let* $g \geq 2$ *and* $F \in$ *Map* S_g *be a pseudo-Anosov mapping class. Let* M_n *be the mapping torus of* F^n. *Then there exists some* n_F *such that for all* $n \geq n_F$

$$\text{rank } \pi_1(M_n) = 2g + 1$$

and every minimal generating tuple is Nielsen equivalent to the standard generating tuple.

Some remarks:

1. The proof is much more brute force than the Heegaard genus analogue of Bachmann-Schleimer, Lackenby, Rubinstein. It uses Zieschang's uniqueness result.
2. The analogous result for mapping tori of free groups with a generic automorphism can be proven the same way.

Namazi proves uniqueness of generating tuples and Heegaard splittings for manifolds given by a Heegaard splitting with large handlebody distance and a technical condition called R-bounded combinatorics.

Agol uses the existence of small graphs representing the fundamental group (Delzant) in his work on rank=2 versus genus=2. Souto uses a variation of this for the case $n = 3$. Also for $n \geq 3$?

The rank problem

If one knows that any generating set consists of short elements after Nielsen equivalence it follows that one only needs to check tuples of short length.

Thus one needs to be able to solve the generalized word problem (membership problem) to solve the rank problem. This is trivial for locally quasiconvex hyperbolic groups and follows from tameness for Kleinian groups.

Theorem 11 (Kapovich,W.) *There exists an algorithm that computes the rank of a given word-hyperbolic Kleinian group or locally quasi-convex word-hyperbolic group.*

Note that the rank problem for word-hyperbolic groups is not decidable. This was shown by Baumslag, Miller und Short using the Rips construction.

Foldings

Introduced by Stallings to study the subgroup structure of free groups. Allows beautiful simple proofs of many old and new results.

Variations of foldings were developed by Bestvina-Feighn, Dunwood, Kapovich-Myasnikov-W, White, Arzhantseva ...

Let G be a group. A *G-graph* $\mathcal{B}$ is a graph of groups $\mathbb{B}$ with base point v_0 such that the following hold:

1. Every vertex group B_v is a subgroup of G.
2. Every edge $e \in EB$ is labelled by an element g_e and $g_{e^{-1}} = g_e^{-1}$.
3. For every edge $e \in EB$ we have $\alpha_e(c) = g_e \omega_e(c) g_e^{-1}$ for all $c \in B_e$.

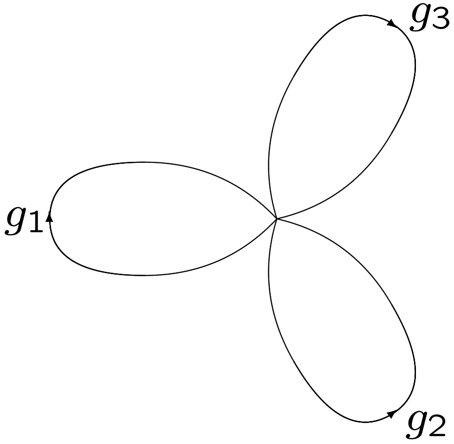
In the case of a torsion-free word-hyperbolic group all edge groups of $\mathbb{B}$ will be trivial, $\pi_1(\mathbb{B})$ is therefore a free product.

Lemma 12 *For any G-graph $\mathcal{B}$ we have that the map $\nu : \pi_1(\mathbb{B}, v_0) \to G$ given by*

$$[b_0, e_1, b_1, \ldots, b_{k-1}, e_k, b_k] \mapsto b_0 g_{e_1} b_1 \cdot \ldots \cdot b_{k-1} g_{e_k} b_k$$

is a homomorphism. We call the subgroup $\nu(\pi_1(\mathbb{B}, v_0))$ of G the subgroup represented by $\mathcal{B}$.

For any generating tuples $(g_1, \ldots, g_n)$ one might just take the wedge of of n circles with labels of the edges the g_i.

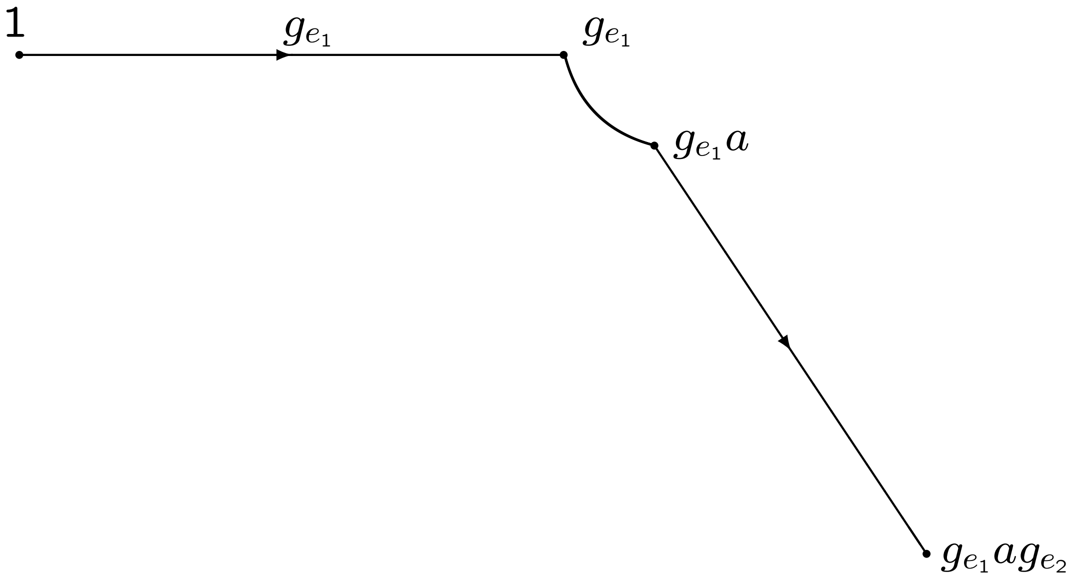


The aim of foldings is to modify $\mathcal{B}$ such that ν is injective. To do this it suffices to show that reduced words get mapped to local quasi-geodesics as those are then global quasi-geodesics and therefore not closed.

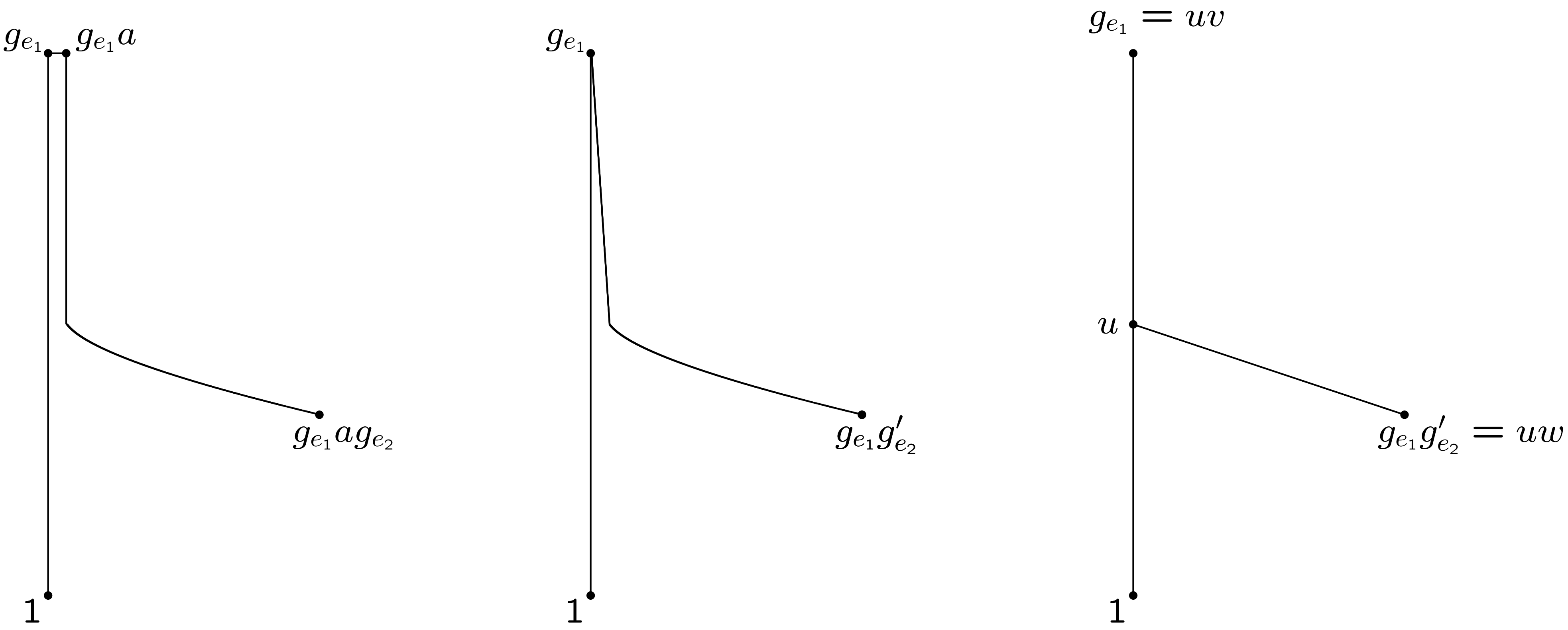
For simplicity we assume that all vertex groups are quasi-convex. We may assume that $|g_e| \leq |g_e a|$ for all $a \in B_{\omega(e)}$ as we otherwise replace g_e with $g_e a$. Now the edges are perpendicular to the vertex groups.

It suffices to study reduced $\mathbb{B}$-paths of type $1, e_1, a, e_2, 1$

If $|a|$ is big then all works out:



If $|a|$ is small then and there is no quasi-geodesic then we can assume that $a = 1$ after replacing g_{e_2} with $g'_{e_2} = ag_{e_2}$ and then fold as in the figure to obtain a smaller G-graph.



Here u, v, w are almost geodesics with $g_{e_1} = uv$ and $g'_{e_2} = v^{-1}w$.

Vertex groups B_v are assume to be generated by tuples $\mathcal{T}_v$ of short elements. If there exists a short loop in B then the element generated by the loop is added to a vertex tuple near the loop.

If two vertices with non-trivial tuples are close they get joined.

The following follows from the discussion:

1. If the G-graph $\mathcal{B}$ is folded, i.e. no further of these modifications can be made then the map $\mu : \pi_1(\mathbb{B}) \to G$ is injective.
2. If $Im(\mu)$ is not a free product then $\mathbb{B}$ must consist of a single vertex.
3. All elements in the generating tuple of this vertex are short.

Uniqueness of folded objects

The folded object appears to depend on the choice of the folding sequence. Does it really?

Conjecture 13 (Dunwoody, Zimmermann) *Let* $H = G * \langle z \mid - \rangle \cong G * \mathbb{Z}$ *and* $\mathcal{T} = (h_1, \ldots, h_k)$ *and* $\mathcal{T} = (h'_1, \ldots, h'_k)$ *be generating tuples of* G.

Then $\mathcal{T}$ *is Nielsen equivalent to* $\mathcal{T}'$ *iff* $(h_1, \ldots, h_k, z)$ *is Nielsen equivalent to* $(h'_1, \ldots, h'_k, z)$

In the case of irreducible tuples the above conjecture is a special case of the following:

Theorem 14 (W) *Let* $G = H * K$, $\mathcal{T} = (h_1, \ldots, h_l, k_1, \ldots, k_m)$ *and* $\mathcal{T}' = (h'_1, \ldots, h'_{l'}, k'_1, \ldots, k'_{m'})$ *with* $h_i, h'_i \in H$ *and* $k_i, k'_i \in K$ *for all* i.

If $\mathcal{T}$ *and* $\mathcal{T}'$ *are irreducible and Nielsen equivalent then* $k = k'$, $l = l'$, $(h_1, \ldots h_k)$ *is Nielsen equivalent to* $(h'_1, \ldots, h'_k)$ *and* $(k_1, \ldots k_l)$ *is Nielsen equivalent to* $(k'_1, \ldots, k'_l)$.

1. This in algebraic analogue of the Gordon conjecture on Heegaard splittings of connected sums, now proven by Bachman and Qiu.

2. Theorem 14 does not hold for reducible tuples. To prove Conjecture 13 the setup has to be changed.

3. For cyclic free factors this is a theorem of Lustig and Moriah.

The basic idea of the proof is the same as in the proof of the following:

Theorem 15 *Let* $F_n = F(x_1, \ldots, x_n)$ *and* $g \in F_n$. *Then there exists a unique reduced word representing* g.

The proof has 2 steps:

1. Step: Show that either there is a unique reduced word representing g or there is a word w and two reduced words w_1 and w_2 that can both be obtained from w by elementary reductions.

2. Step: Choose such a w of minimal length and derive a contradiction using the diamond lemma:

$$\begin{tikzcd}
 & ux_1x_1^{-1}vx_2^{-1}x_2w \arrow[dl] \arrow[dr] & \\
uvx_2^{-1}x_2w \arrow[dr] & & ux_1x_1^{-1}vw \arrow[dl] \\
 & uvw &
\end{tikzcd}$$

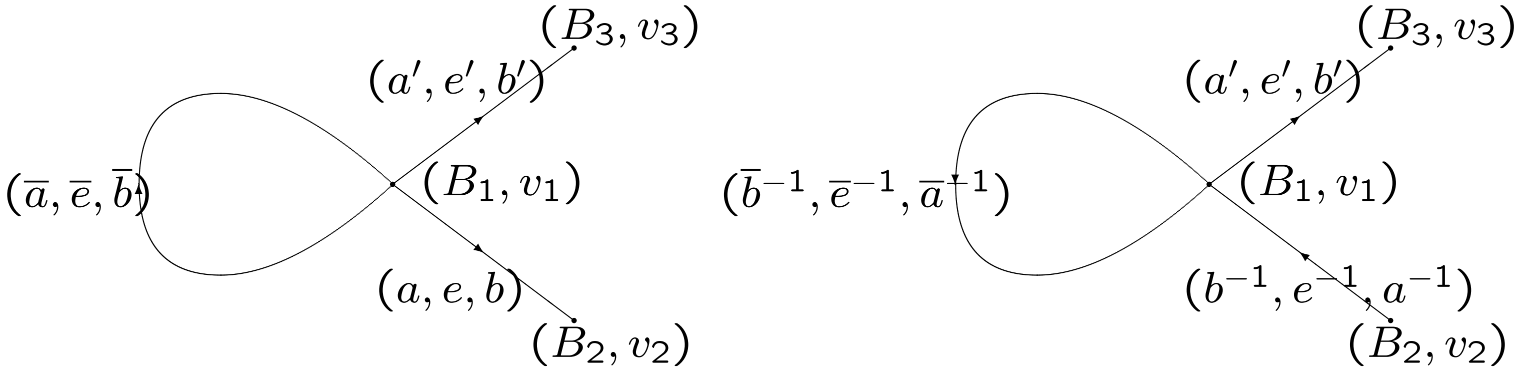
Definition $\mathbb{A}$-graph An $\mathbb{A}$-graph $\mathcal{B}$ consists of an underlying graph B with the following additional data:

1. A graph-morphism $[\,.\,] : B \to A$.

2. For each $u \in VB$ there is a group B_u with $B_u \leq A_{[u]}$.

3. To each edge $f \in EB$ there are two associated group elements $f_\alpha \in A_{[\alpha(f)]}$ and $f_\omega \in A_{[\omega(f)]}$ such that $(f^{-1})_\alpha = (f_\omega)^{-1}$ for all $f \in EB$.

Two distinct diagrams associated to the same $\mathbb{A}$-graph:



Let $\mathbb{B}$ be the graph of groups corresponding to $\mathcal{B}$.

Let $p = b_0, f_1, b_1, \ldots, f_s, b_s$ be an $\mathbb{B}$-path. Each edge f_i has a label (g_i, e_i, k_i) in $\mathcal{B}$.

To p we associate the $\mathbb{A}$-path $\mu(p)$ given by

$$(b_0 g_1), e_1, (k_1 b_1 g_2), e_2, \ldots, (k_{s-1} b_{s-1} g_s), e_s, (k_s b_s).$$

Proposition 16 *Let $\mathcal{B}$ be an $\mathbb{A}$-graph, $u_0 \in VB$ and $v_0 = [u_0]$. Then:*

1. *If $p \sim p'$ as $\mathbb{B}$-paths, then $\mu(p) \sim \mu(p')$ as $\mathbb{A}$-paths.*
2. *The map μ restricted to the set of $\mathbb{B}$-paths from u_0 to u_0 factors through to a homomorphism $\nu : \pi_1(\mathbb{B}, u_0) \to \pi_1(\mathbb{A}, v_0)$.*

Let $\mathcal{B}$ be an $\mathbb{A}$-graph. We will say that $\mathcal{B}$ is *folded* if the following hold:

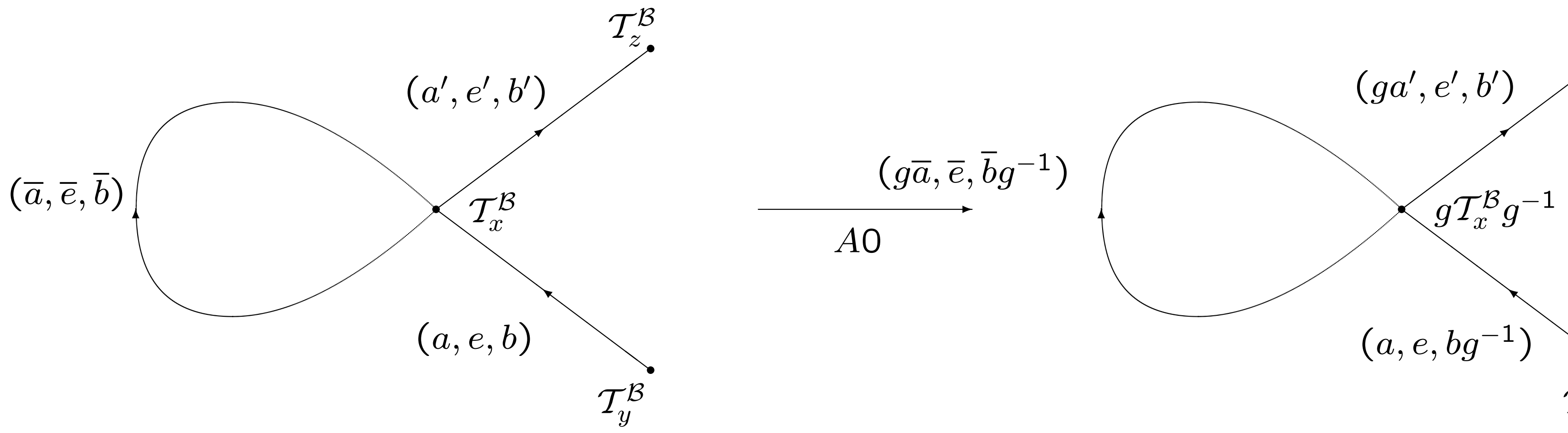
For any two distinct edges $f_1, f_2 \in EB$ with $\alpha(f_1) = \alpha(f_2) = z$ and labels (a_1, e, b_1), (a_2, e, b_2) we have $a_2 \neq a'a_1$ for all $a' \in B_z$.

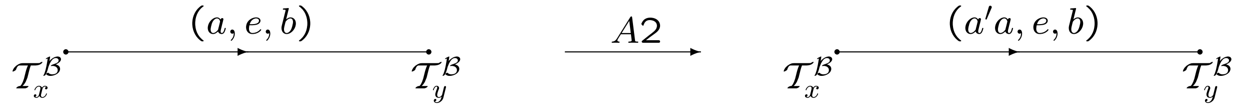
Proposition 17 *Let $\mathcal{B}$ be a folded $\mathbb{A}$-graph defining the graph of groups $\mathbb{B}$. Let $U = \nu(\pi_1(\mathbb{B}, u_0)) \leq G = \pi_1(\mathbb{A}, v_0)$. Then the following hold:*

1. *For any reduced $\mathbb{B}$-path p the corresponding $\mathbb{A}$-path $\mu(p)$ is $\mathbb{A}$-reduced.*
2. *The epimorphism $\nu : \pi_1(\mathbb{B}, u_0) \to U$ is an isomorphism.*
3. *$\mathbb{B}$ is isomorphic to the induced splitting of U as a subgroup of $\pi_1(\mathbb{A}, v_0)$.*
4. *If $U = \pi_1(\mathbb{A}, v_0)$ then $\mathbb{A} \cong \mathbb{B}$.*

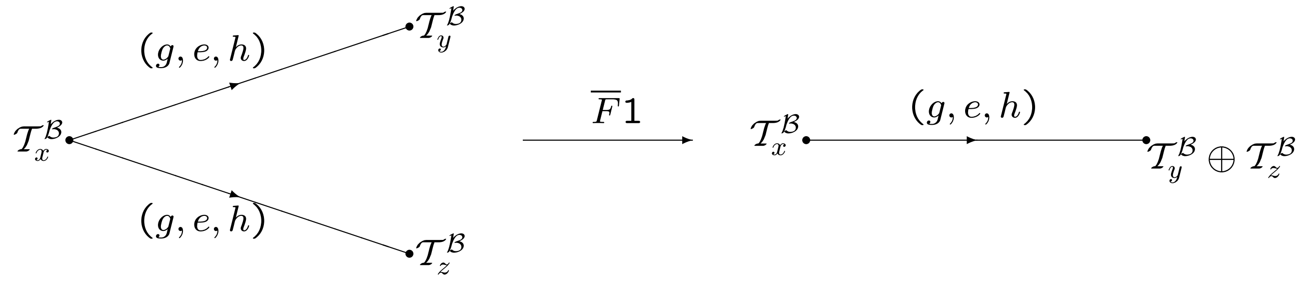
Call now an $\mathbb{A}$-graph $\mathcal{B}$ a marked $\mathbb{A}$-graph if to any vertex group B_v there is an associated generating tuples $\mathcal{T}_v^{\mathcal{B}}$

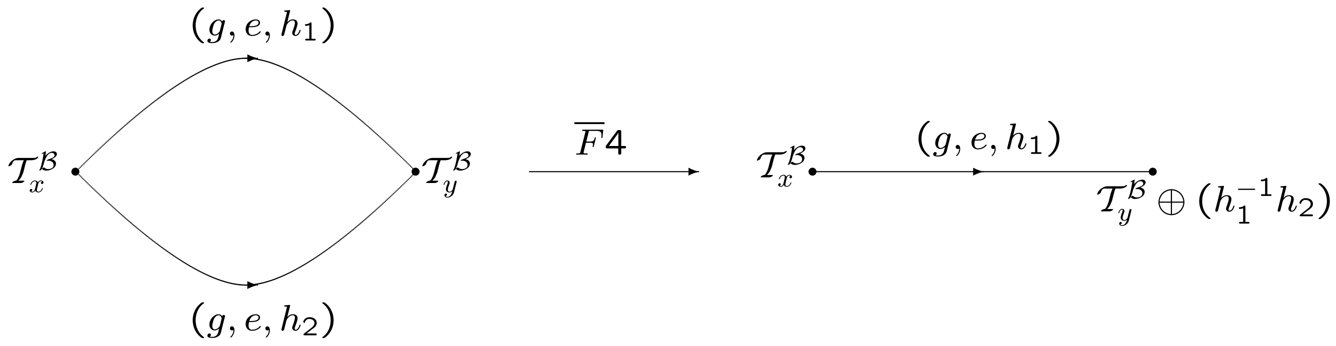
Auxiliary moves, $g \in A_{[x]}$ **and** $a' \in B_x = \langle \mathcal{T}_x^{\mathcal{B}} \rangle$**:**





Elementary folds:





To any $\mathbb{A}$-graph $\mathcal{B}$ with base vertex u_0 and maximal tree T we can associate in a natural way a generating tuple $\mathcal{T}_T^{\mathcal{B}}$ of $\pi_1(\mathbb{B}, u_0)$ as follows:

1. For any pair $e, e^{-1} \in EB$ take the element $[\gamma_{\alpha(e)}, e, \gamma_{\omega(e)^{-1}}]$.
2. For any $g \in \mathcal{T}_v^{\mathcal{B}}$ take $[\gamma_v \cdot g \cdot \gamma_v^{-1}]$.

For $v \in VB$ γ_v the unique reduced $\mathbb{B}$-path $1, e_{v1}, 1, \ldots, 1, e_{vm_v}, 1$ from u_0 to v in T.

Lemma 1 If T, T' are maximal subtrees of B, then $\mathcal{T}_T^{\mathcal{B}} \sim \mathcal{T}_{T'}^{\mathcal{B}}$, thus there exists a well-defined Nielsen equivalence class $[\mathcal{T}^{\mathcal{B}}]$.

Thus to any marked $\mathbb{A}$-graph $\mathcal{B}$ there is an associated Nielsen equivalence class of generating tuples of $U = \nu(\pi_1(\mathbb{B}, u_0))$, namely $\nu([\mathcal{T}^{\mathcal{B}}])$.

Two marked $\mathbb{A}$-graphs are called *equivalent* if one can be obtained from the other by finitely many modifications of one of the following types:

1. Replace the generating tuple of some vertex group by some Nielsen equivalent tuple.
2. Apply some auxiliary move.

Lemma 2 If $\mathcal{B}, \mathcal{B}'$ are equivalent marked $\mathbb{A}$-graphs then $\nu([\mathcal{T}^{\mathcal{B}}]) = \nu([\mathcal{T}^{\mathcal{B}'}])$. Thus there exists a well-defined Nielsen equivalence class $\nu([\mathcal{T}^{[\mathcal{B}]}])$.

Folds also preserve the Nielsen equivalence class.

Lemma 3 Let $\mathcal{B}, \mathcal{B}'$ be marked $\mathbb{A}$-graphs such that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by an elementary fold. Then $\nu([\mathcal{T}^{\mathcal{B}}]) = \nu([\mathcal{T}^{\mathcal{B}'}])$.

Let $\mathbb{A}$ be the graph of groups consisting of a single edge with trivial edge group and vertex groups H and K. Thus $\pi_1(\mathbb{A}) = H * K$.

Let $\mathcal{T}$ be a generating tuple of $G = \pi_1(\mathbb{A}) = H * K$. The directed $\Gamma_{[\mathcal{T}]}$ is then defined as follows:

1. The vertices are the equivalence classes of marked $\mathbb{A}$-graphs ω with $\nu([\mathcal{T}^{\omega}]) = [\mathcal{T}]$.
2. There is a direct edge from ω to ω', if there exist $\mathcal{B} \in \omega$ and $\mathcal{B}' \in \omega'$ such that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by an elementary fold. We write $\omega \mapsto \omega'$.

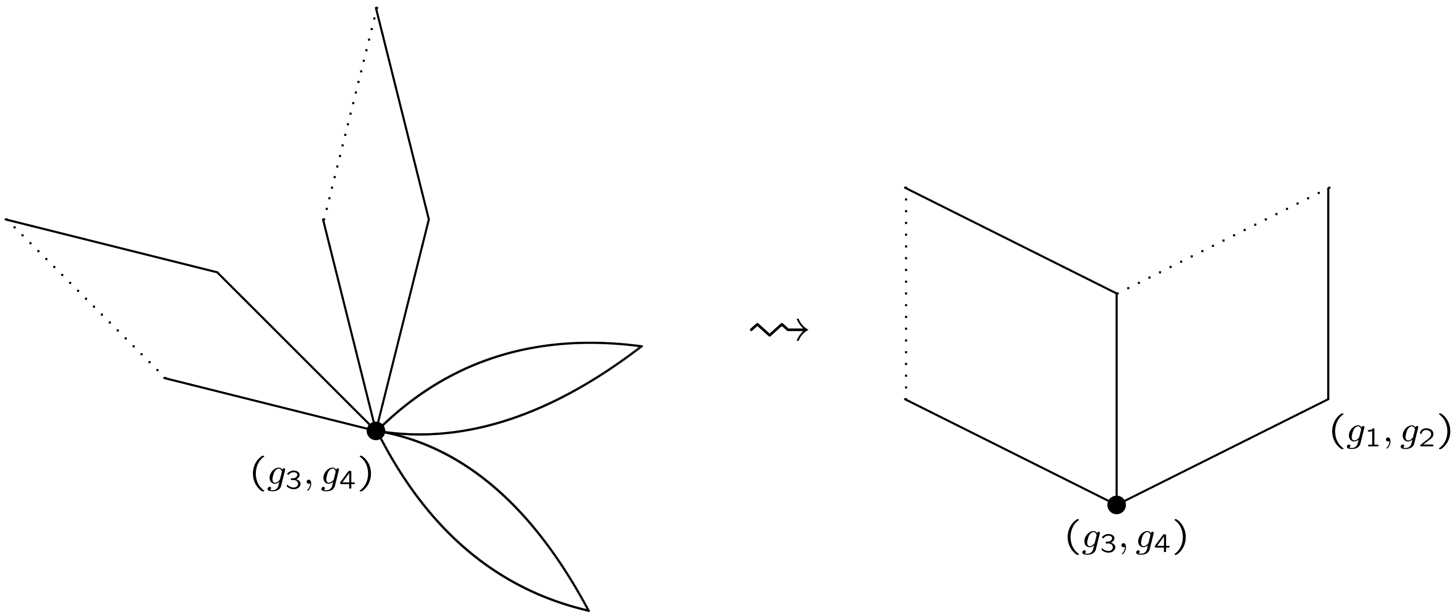
We denote the number of vertices of the underlying graph of ω by $H(\omega)$, the height of ω.

Note that Grushko's theorem states that $\Gamma_{[\mathcal{T}]}$ contains a vertex of height one and the assertion of our theorem is that this vertex is unique.

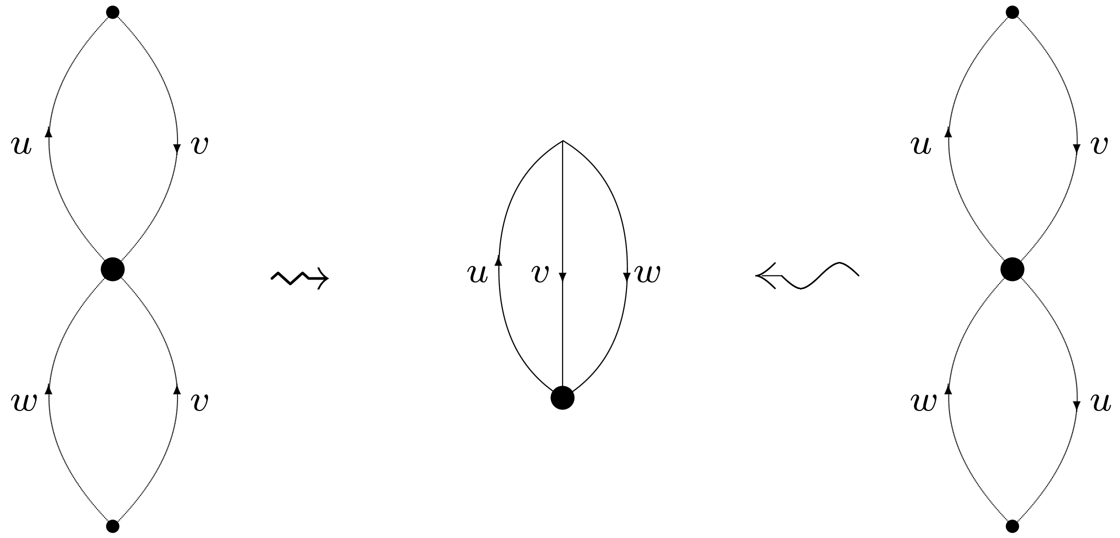
Lemma 4 $\Gamma_{[\mathcal{T}]}$ is connected.

Let $\omega, \omega' \in V\Gamma_{[\mathcal{T}]}$. We have to show that there exists a (not necessarily directed) path from ω to ω'. The proof has two steps:

Step 1: Unfolding into a wedge of circles gives a new marked $\mathbb{A}$-graph in the same component of $\Gamma_{[\mathcal{T}]}$.



Step 2: An elementary equivalence involving two generators is possible within one component.



The figure depicts the elementary equivalence from (g, h) to (g, gh) where g is represented by the $\mathbb{A}$-path uv, h by the $\mathbb{A}$-path $v^{-1}w$ and gh by the $\mathbb{A}$-path uw.

The wedges of two circle representing (g, h) and (g, gh) project onto the same element and lie therefore in the same component.

It remains to verify the diamond Lemma.

Lemma 5 Suppose that T is irreducible, that $\omega, \omega_1, \omega_2 \in V\Gamma_{[\mathcal{T}]}$ such that $\omega \mapsto \omega_1$ and $\omega \mapsto \omega_2$. Then there exists some ω' such that $\omega_1 \mapsto \omega'$ and $\omega_2 \mapsto \omega'$.

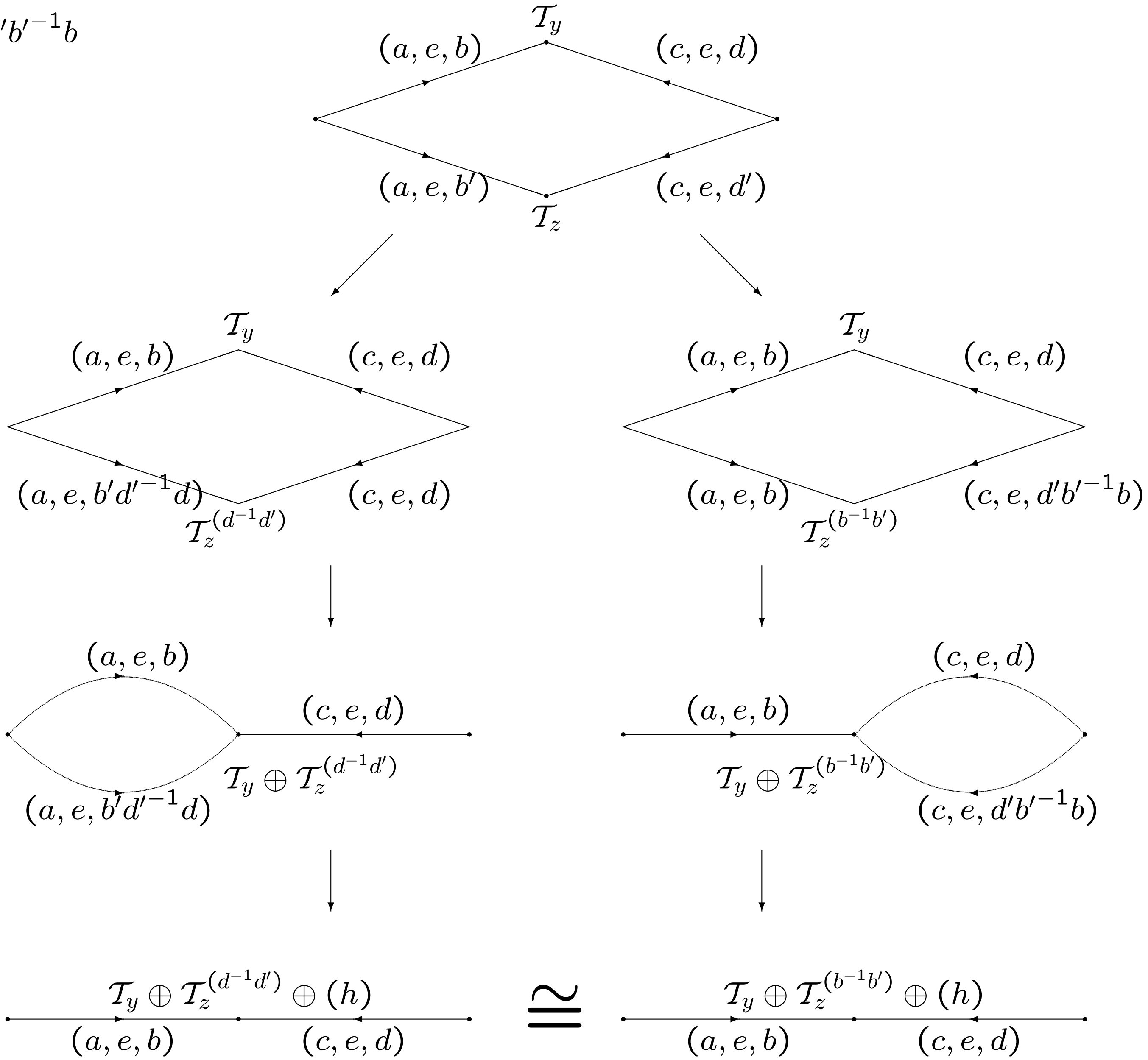
Due to the local nature of the folds this is trivial in most cases as the subgraphs involved in the different folds are disjoint.

There are essentially two cases to worry about:

1. The subgraphs involved in the folds are different but not disjoint.

2. The subgraphs involved are the same.

$h := d^{-1}d'b'^{-1}b$


$\mathcal{T}_y$
(a,e,b)
(c,e,d)
(a,e,b')
(c,e,d')
$\mathcal{T}_z$
$\mathcal{T}_y$
(a,e,b)
(c,e,d)
$(a,e,b'd'^{-1}d)$
(c,e,d)
$\mathcal{T}_z^{(d^{-1}d')}$
$\mathcal{T}_y$
(a,e,b)
(c,e,d)
(a,e,b)
$(c,e,d'b'^{-1}b)$
$\mathcal{T}_z^{(b^{-1}b')}$
(a,e,b)
(c,e,d)
$\mathcal{T}_y \oplus \mathcal{T}_z^{(d^{-1}d')}$
$(a,e,b'd'^{-1}d)$
(c,e,d)
(a,e,b)
$\mathcal{T}_y \oplus \mathcal{T}_z^{(b^{-1}b')}$
$(c,e,d'b'^{-1}b)$
$\mathcal{T}_y \oplus \mathcal{T}_z^{(d^{-1}d')} \oplus (h)$
(a,e,b)
(c,e,d)
$\cong$
$\mathcal{T}_y \oplus \mathcal{T}_z^{(b^{-1}b')} \oplus (h)$
(a,e,b)
(c,e,d)

If the involved subgraphs are identical the folds cannot commute as the loop disappears in the fold. It turns out however that in this case the tuple must have been reducible.