Random Dirichlet Polynomials: Metric Entropy Approach

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1. Random processes and metric entropy

Random process:

$$X(t), \qquad t \in T.$$

Natural distance on T:

$$\rho(s,t)^2 = \mathbb{E}(X(s) - X(t))^2.$$

Covering numbers and metric entropy

$$N(\varepsilon) = \inf\{n : \exists A_1 \dots A_n : T \subset \cup_1^n A_j,\ diam A_j \le \varepsilon\}.$$

$$H(\varepsilon) = \log N(\varepsilon).$$

Main idea:

Size of X $\leftrightarrow$ growth of $H(\varepsilon)$ as $\varepsilon \to 0$.

2. Basic results using metric entropy

Let X be centered: $\mathbb{E}X(t) = 0$ and Gaussian: $X(t)$ has a normal (Gaussian) distribution.

We measure the size of X with $\mathbb{E}\sup_{t\in T} X(t)$. Then:

a) *Sudakov–Fernique lower bound*

$$\mathbb{E}\sup_{t\in T} X(t) \geq C\ \varepsilon\ H(\varepsilon)^{1/2} \qquad \forall\varepsilon > 0.$$

($C = 1.5$)

b) *Dudley upper bound*

$$\mathbb{E}\sup_{t\in T} X(t) \leq C \int_0^\infty H(\varepsilon)^{1/2} d\varepsilon$$

($C = 4\sqrt{2}$)

3. Dirichlet series and polynomials

Dirichlet series:

$$\varphi(t) = \sum_{n=1}^{\infty} a_n \, e^{it\lambda_n}, \qquad t \in \mathbb{R}.$$

Dirichlet polynomials:

$$\mathcal{P}_N(t) = \sum_{n=1}^{N} a_n \, e^{it\lambda_n}, \qquad t \in \mathbb{R}.$$

Trigonometric series and polynomials (Fourier theory): $\lambda_n = n$.

Classical Dirichlet series and polynomials: $\lambda_n = \log n$.

$$\varphi(t) = \sum_{n=1}^{\infty} a_n \, n^{it}, \qquad t \in \mathbb{R}.$$

Example: Riemann zeta function $a_n = n^{-\sigma}$,

$$\varphi(t) = \sum_{n=1}^{\infty} n^{-\sigma+it}, \qquad t \in \mathbb{R}.$$

General task: extend Fourier theory to Dirichlet case.

A small subtask: Rudin-Shapiro polynomials

Consider a trigonometric polynomial:

$$\mathcal{P}(t) = \sum_{n=0}^{N} a_n \, e^{itn}, \qquad 0 \le t \le 2\pi.$$

Then,

$$\|\mathcal{P}\|_\infty = \sup_t |\mathcal{P}(t)| \le \sum_n |a_n|.$$

Conversely,

$$\|\mathcal{P}\|_\infty^2 \ge \frac{1}{2\pi} \int_0^{2\pi} |\mathcal{P}(t)|^2 dt = \sum_n |a_n|^2 \ge \left(\frac{\sum_n |a_n|}{\sqrt{N}} \right)^2 .$$

Thus

$$\|\mathcal{P}\|_\infty \ge \frac{\sum_n |a_n|}{\sqrt{N}} .$$

Rudin and Shapiro: $\exists (a_n) \in \{\pm 1\}$:

$$\|\mathcal{P}\|_\infty \sim (2 + \sqrt{2}) \sqrt{N} \approx \frac{\sum_n |a_n|}{\sqrt{N}} .$$

These $\mathcal{P}$ are *small deviation* polynomials.

What about small deviation Dirichlet polynomials ? ($-\infty < t < \infty$). Konyagin (Moscow) and Queffelec (Lille) proved:

a) $\exists c_1, \beta_1 :\ \forall N, \forall (a_n)$

$$\|\mathcal{P}\|_\infty \geq \frac{c_1 \ \sum_n |a_n|}{\sqrt{N}} \ \exp\left(\beta_1 \sqrt{\log N \cdot \log\log N}\right) \ .$$

b) $\exists c_2, \beta_2, N_0 :\ \forall N \geq N_0, \exists\, \mathcal{P}_N = \sum_{n=1}^N a_n n^{it}$:

$$\|\mathcal{P}\|_\infty \leq \frac{c_2 \ \sum_n |a_n|}{\sqrt{N}} \ \exp\left(\beta_2 \sqrt{\log N \cdot \log\log N}\right) \ .$$

This is attained by choosing *random* coefficients and studying $\mathbb{E}\|\mathcal{P}_N\|_\infty$.

Rudin-Shapiro bound is not attained.

No explicit (a_n) is known.

Complicated answer.

5. Size of random Dirichlet polynomials

Take the simplest classical random Dirichlet polynomials

$$\mathcal{P}_N(t,\omega) = \sum_{n=1}^{N} n^{-\sigma} \varepsilon_n(\omega) n^{it}$$

with ε_n independent standard normal or symmetric Bernoulli (± 1).

Theorem 1 *(Halász, Queffelec) For any positive $\sigma < 1$*

$$M_N := \mathbb{E}\|\mathcal{P}_N(t,\omega)\|_\infty \approx \frac{N^{1-\sigma}}{\log N}, \qquad N \to \infty.$$

Since $\sum_{n=1}^N |a_n| \approx N^{1-\sigma}$, we see that M_n is smaller than $\sum_{n=1}^N |a_n|$ but much larger than $\frac{\sum_{n=1}^N |a_n|}{\sqrt{N}}$. These random polynomials are not the small deviation ones.

We will consider somewhat more tricky polynomials.

Let (p_τ) be the sequence of all prime numbers.

Denote $P^+(n)$ the largest prime divisor of n.

Denote

$$\mathcal{V}_{\tau,N} = \left\{ n : n \le N, P^+(n) \le p_\tau \right\} \subset [1; N]$$

and consider random polynomial

$$\mathcal{P}_{\tau,N}(t) = \sum_{n \in \mathcal{V}_{\tau,N}} n^{-\sigma} \varepsilon_n n^{it}$$

as a candidate for being a small deviation polynomial. Then

$$M_{\tau,N} = \mathbb{E} \sup_{t\in\mathbb{R}} |\mathcal{P}_{\tau,N}(t)|$$

should be compared with $\sum_{n\in\mathcal{V}_{\tau,N}} n^{-\sigma}$.

Important: $a_n = n^{-\sigma}$ satisfy $a_{mn} = a_m a_n$.

6. Number theory: Dickman function.

In order to understand $\sum_{n \in \mathcal{V}_{\tau,N}} n^{-\sigma}$, let $\sigma = 0$ and consider $\left|\mathcal{V}_{\tau,N}\right| = \left|\{n : P^{+}(n) \leq p_{\tau}, n \leq N\}\right|$. This is related to Dickman function.

If $M \to \infty$, $N \to \infty$ so that

$$\frac{\log N}{\log M} \to u \qquad (M \approx N^{1/u}),$$

then

$$\frac{\left|\{n : P^{+}(n) \leq M, n \leq N\}\right|}{N} \to \rho(u).$$

(there is a differential equation but no explicit formula for $\rho(u)$).

Therefore, if $\tau \approx N^{1/u}$, then $\left|\mathcal{V}_{\tau,N}\right| \sim \rho(u)N$.

7. Main result Recall that

$$M_{\tau,N} = \mathbb{E} \sup_{t\in\mathbb{R}} |\mathcal{P}_{\tau,N}(t)|$$

Theorem 2 *For any $\sigma < 1$, the average $M_{\tau,N}$ satisfies:*

$$M_{\tau,N} \leq \begin{cases} C\ \frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}}, & N^{1/2} \leq \tau \leq N, \\ C\ \frac{N^{3/4-\sigma}}{(\log N)^{1/2}}, & \frac{N^{1/2}}{\log N} \leq \tau \leq N^{1/2}, \\ C\, N^{1/2-\sigma}\tau^{1/2}\rho\left(\frac{N}{p_\tau}\right)^{1/2}, & \tau \leq \frac{N^{1/2}}{\log N} \end{cases}$$

and

$$M_{\tau,N} \geq C\ \frac{N^{1/2-\sigma}\tau^{1/2}}{(\log \tau)^{1/2}}\ \rho\left(\frac{N}{p_\tau}\right)^{1/2}.$$

Remark: in the first zone the estimates are order-optimal. In other zones a log-gap appears. First zone contains Halász-Queffelec estimate.

Open problem: find the size asymptotics of random Dirichlet polynomials

$$\mathbb{E} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n \varepsilon_n n^{it}(t) \Big| \approx ???$$

for suitable class of multiplicative (a_n).

The upper bound in Theorem 2 with the right choice of τ,

$$\tau \approx \exp\left\{ \sqrt{\frac{\log N}{2} \log\log N} \right\}$$

yields (as by Konyagin and Queffelec)

$$\|\mathcal{P}_{\tau,N}\|_\infty \le \frac{c_2 \ \sum_n |a_n|}{\sqrt{N}} \ \exp\left(\beta_2 \sqrt{\log N \cdot \log\log N}\right) \ .$$

but with a better (maybe optimal) parameter β_2.

8. Proof idea: upper bound via metric entropy

Step 1: Harald Bohr's reduction: from a random polynomial on the line to a random function on the high-dimensional torus. Let $\mathbb{S}$ be the unit circle,

$$\pi(N) = |\{p: \ p \le N, p \text{ is prime}\}|.$$

Consider the time change application f,

$$f : \mathbb{R} \to \mathbb{S}^{\pi(N)}, \qquad f(t) = \left(p_1^{it}, \ldots, p_{\pi(N)}^{it}\right).$$

Then the image $f(\mathbb{R})$ is dense in $\mathbb{S}^{\pi(N)}$.

We use the canonical representation,

$$n = \prod_{j \le \pi(N)} p_j^{b_j(n)}.$$

and obtain for any polynomial

$$\mathcal{P}_N(t) = \sum_{n=2}^{N} a_n n^{it} = \sum_{n=2}^{N} a_n \left(\prod_{j \le \pi(N)} p_j^{b_j(n)} \right)^{it}$$
$$= \sum_{n=2}^{N} a_n \prod_{j \le \pi(N)} \left(p_j^{b_j(n)} \right)^{it}$$
$$= \tilde{\mathcal{P}}(f(t)),$$

where

$$\tilde{\mathcal{P}}(z) = \sum_{n=2}^{N} a_n \prod_{j \le \pi(N)} z_j^{b_j(n)}.$$

Hence,

$$\sup_{t \in \mathbb{R}} |\mathcal{P}(t)| = \sup_{z \in \mathbb{S}^{\pi(N)}} |\tilde{\mathcal{P}}(z)|.$$

Step 2 : choose a clever covering of the torus $\mathbb{S}^{\pi(N)}$ and use Dudley estimate....

9. Lower bound: restriction of the process

First, restriction the of parameter set. We have to evaluate

$$\mathbb{E} \sup_{z\in\mathbb{S}^{\pi(N)}} \left| \sum_{n\in\mathcal{V}_{\tau,N}} \varepsilon_n \, n^{-\sigma} \prod_{j\le\pi(N)} z_j^{b_j(n)} \right|$$

$$= \mathbb{E} \sup_{z\in\mathbb{S}^{\tau}} \left| \sum_{n\in\mathcal{V}_{\tau,N}} \varepsilon_n \, n^{-\sigma} \prod_{j\le\tau} z_j^{b_j(n)} \right|, \qquad \varepsilon_n = \pm 1.$$

$$\ge \mathbb{E} \sup_{z\in\mathcal{Z}} \left| \sum_{n\in\mathcal{V}_{\tau,N}} \varepsilon_n \, n^{-\sigma} \prod_{\tau/2\le j\le\tau} z_j^{b_j(n)} \right|,$$

where

$$\mathcal{Z} = \{z \in \mathbb{S}^\tau : z_j = 1, j \le \tau/2; z_j = \pm 1, j > \tau/2\}.$$

Next, restriction of the process: keep only special terms. Let for $\tau/2 < j \le \tau$,

$$L_j = \{n \in \mathcal{V}_{\tau,N} : \ n = p_j\, \tilde{n}, \quad P_+(\tilde{n}) \le p_{\tau/2}\}.$$

For $n \in L_j$ we have $b_j(n) = 1$ and $b_i(n) = 0$ elsewhere. We have

$$\mathbb{E} \sup_{z\in\mathcal{Z}} \left| \sum_{n\in\mathcal{V}_{\tau,N}} \varepsilon_n\, n^{-\sigma} \prod_{\tau/2\le j\le\tau} z_j^{b_j(n)} \right|$$

$$\ge \mathbb{E} \sup_{z\in\mathcal{Z}} \left| \sum_{\tau/2<j\le\tau} \quad \sum_{n\in L_j} \varepsilon_n\, n^{-\sigma} \prod_{\tau/2\le j\le\tau} z_j^{b_j(n)} \right|$$

$$= \mathbb{E} \ \sup_{z\in\mathcal{Z}} \left| \sum_{\tau/2<j\le\tau} \quad \sum_{n\in L_j} \varepsilon_n\, n^{-\sigma} z_j \right|$$

$$= \sum_{\tau/2<j\le\tau} \mathbb{E} \left| \sum_{n\in L_j} \varepsilon_n\, n^{-\sigma} \right| \approx \sum_{\tau/2<j\le\tau} \left| \sum_{n\in L_j} n^{-2\sigma} \right|$$

$$\ldots \ge \frac{\tau^{1/2} N^{1/2-\sigma}}{(\log \tau)^{1/2}} \ . \qquad \Box$$