

Workshop on Optimal Transportation, Transport Equations, and Hydrodynamics

Of the 28 participants named in the proposal, 20 were able to attend, and these were supplemented by 2 more chosen by the organisers, and 8 others who asked for invitations. The participants included 3 research students, and 5 postdocs or recent PhDs. There were 9 based in UK institutions, 14 from continental Europe, and 7 from the Americas; the total of 30 included 4 women.

Cordero-Erausquin gave two 60-minute overview lectures on Optimal Transportation, concentrating on applications to geometric inequalities on manifolds. In particular, he described the extension of the Prékopa-Leindler inequality, a counterpart for functions to the classical Brunn-Minkowski inequality of convex geometry, to the setting of Riemannian manifolds of positive curvature.

In addition, there were 21 invited or contributed lectures of 45 minutes duration. The lectures were timetabled for the morning and late afternoon, with an extended break for lunch and informal mathematical activity between; on Wednesday there was a free afternoon, and on Friday a reduced midday break.

For background in transport equations, Ambrosio emailed 72 pages of recent lecture notes, which were circulated at the workshop. De Lellis spoke on quantitative estimates for stability of transport equations, and compactness for flows of transport equations.

One dominant theme in Hydrodynamics was Semigeostrophic Theory (SG), developed to model large-scale vortex phenomena in the atmosphere and ocean. Feldman described the role of Ambrosio's work on transport equations in passing between Euclidean SG and a dual formulation. It is physically most realistic to take the domain to be spherical, and Cullen described the modelling underlying SG on the sphere. Douglas described theoretical work in progress on Brenier's polar factorisation, motivated by front formation in SG. A second theme was the theory of generalised fluid flows, which are a relaxation, due to Brenier, of Arnol'd's formulation of hydrodynamics. They are well suited to address the optimal incompressible transport problem in three space dimensions, but not in two space dimensions, for reasons that go back to Shnirelman's seminal paper in the mid 80s. In his lecture, Shnirelman introduced a notion of *generalised flow with definite velocity*, to deal with the two dimensional case. By adding topological constraints and prescribing the braid structure of all finite collections of particle trajectories, he showed how the optimal generalised fluid flow has indeed a definite velocity field, which is a weak solution to the Euler equations. Lopes studied density-dependent generalised flows. Roulstone described some Riemannian-geometric constructions designed to study conservation of vorticity in ideal fluids. Schochet reviewed recent results on slightly compressible fluids and their incompressible limits, in situations never addressed before, in particular for models including energy equation and periodic boundary conditions for ill-posed initial conditions, which yield delicate averaging techniques and resonance analysis. Mazzucato spoke on the decay of entropy in a dissipative system.

McCann talked about a problem where one measure is to be partially transported to another, subject to a lower bound on the mass transported; this leads to an obstacle problem for a Monge-Ampère equation. Pratelli spoke on developments in the Monge problem with degree-1 homogeneous cost. Buttazzo spoke about optimal configurations for a transport network, where the cost functional contained a continuous term related to private means of transport, and a step-function related to public transport. Carlen spoke about minimisation of a separable integral functional with a double-well potential, and gave an analysis of the

volume-fraction occupied by the two phases using McCann's *displacement-convexity*, which is an application of the Brenier map. Norbury discussed some qualitative results concerning Ginzburg-Landau systems. Carbery gave a talk on Brascamp-Lieb type inequalities, and asked for an optimal transport method for proving them.

Wasserstein distances between probability measures are defined as optimal transportation costs between the measures, with respect to specified cost metrics. Savaré discussed a programme for studying a quantum drift equation as the gradient flow of Fisher information with respect to a Wasserstein distance. Carrillo spoke about contractivity in the 2-Wasserstein distance of the flow of a nonlinear diffusion equation, using displacement convexity. Gangbo considered the equation $\partial u/\partial t = \operatorname{div} \nabla F(\nabla u)$ with quasiconvex F , and gave examples where existence and uniqueness of the Cauchy problem could be proved by reformulating as a gradient flow with respect to a Wasserstein distance. Hauray discussed convergence of finite systems of interacting particles to solutions of the Vlasov equations, assuming that their initial data converge in a Wasserstein metric.

Otto described a proof of a log-Sobolev inequality for probability measures using a 2-scale approach in statistical mechanics. Blower spoke on best constants in log-Sobolev inequalities.

Overall, the workshop was very successful. The format of the meeting worked well, with the participants seizing the opportunity to make new contacts and maintain existing ones; there was an active "after-hours" life as well! The level of attendance remained high even up to the last afternoon. The questionnaire results were overwhelmingly positive, regarding the quality of the talks, the opportunity for discussion, and practical arrangements. We quote three comments concerning the current state of the subject:

"Emphasis seems to be switching from theoretical results about mass transfer to applications of the theory, particularly regarding very weak solutions to systems of PDE."

"Applications to other nonlinear diffusion equations as fourth-order diffusion equations, fluid mechanics versus optimal transport theory has to be further developed."

"Probability in interacting particle systems is likely to be important, as is the use of transportation theory in possibly degenerate PDE."

Yann Brenier
Geoffrey Burton