Applied and computational complex analysis

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Abstracts

Blyth, Mark
Deformation of an elastic cell under inviscid flow
We use complex variable methods to study the nonlinear deformation of a two-dimensional elastic cell placed into a uniform flow. The flow is assumed to be inviscid and irrotational. The problem is governed by two dimensionless parameters which reflect the transmural pressure difference between the interior of the cell and the ambient pressure at infinity, and the strength of the oncoming flow. In the absence of flow, and for sufficiently low transmural pressure across the cell wall, the cell is circular; Flaherty et al (1972) showed that as the transmural pressure increases through different threshold values, buckled states with m-fold rotational symmetry come into play, and that each of these eventually exhibits a point of self-contact at a critical pressure. We focus on the effect of flow on the cell and contrast with the behaviour found for a two-dimensional bubble in a uniform stream. In the second part of the talk we consider the additional effect of circulation around the cell as a simple model of a deformable aerofoil experiencing lift.

Bornemann, Folkmar
Numerical problems inspired by discrete complex analysis
Discrete complex analysis, that is, the quest for discretizing the whole theory and not just single manifestations of it, has led to a wealth of interesting mathematical concepts and a rich nonlinear theory related to integrable systems. A major question is the construction of discrete analogues of the set of "standard" holomorphic maps and a good model problem is the power map z^a. The stable numerical evaluation of that map touches on many mathematical topics ranging from boundary value problems of discrete Painlevé equations to infinite-dimensional linear algebra and discrete optimization.

Cummings, Linda
Slow viscous flows in doubly-connected domains
We discuss the problem of two-dimensional Stokes flow with two free boundaries, at each of which a constant surface tension acts. An integration of the stress boundary conditions leads to (time-dependent) constants of integration associated with each boundary and, while these can be chosen for convenience at one of the free boundaries, they remain arbitrary at the second and must be determined as part of the solution.

Following early work of Hopper and Richardson for the simply-connected problem, and of Crowdy & Tanveer and Richardson for the doubly-connected one (as well as subsequent work by these and other authors), conformal mapping methods from complex analysis are used to reformulate the problem. By means of analytic continuation of the complex form of the boundary conditions at each free boundary, two globally-valid equations governing the motion are obtained: these must be consistent in order for solutions to exist. We discuss the implications of this consistency condition, together with some open questions. Our results are illustrated by means of specific examples.

Dallaston, Michael

Asymptotic selection of self-similar rupture solutions to a generalised thin film equation

This talk concerns similarity solutions to a thin film equation $h_t + (h^m h_{xxx} + h^n h_x)_x = 0$, in which the exponents m and n are arbitrary. Such a model can describe thin film phenomena on different scales, such as rupture due to van der Waals force, destabilisation due to gravity (Rayleigh-Taylor instability), or thermocapillary forces, depending on the exponent n (generally $m = 3$ for a film on a solid substrate).

Computations show that there are a countably infinite number of similarity solutions, which merge via saddle-node bifurcations as n is changed. In this talk we will discuss the asymptotic selection of these solution branches out of a continuum using exponential asymptotic techniques, and the mechanism by which solution branches merge. This extends on a work by Chapman, Trinh & Witelski [SIAM J Appl Math, 2013 (73):232-253], who performed the asymptotic analysis for the special case relevant to van der Waals rupture.

Davis, Anthony

Non existence of further closed form 2-D sloshing modes in a symmetric triangular basin

The study of sloshing modes has a history dating back to Euler. Very few 2-D or 3-D closed form solutions have been constructed and these pertain to the simpler case of inviscid, small amplitude, fluid motion. In 2-D they exist for symmetric channels with vertical walls or straight walls inclined at 45^{circ} or (even modes only) 60^{circ} to the vertical. Fokas' extended transform method facilitates a solution of Laplace's equation in a convex polygon which, for favourable choices of polygon shape and boundary conditions, reduces the solution to a sum of residues in the complex transform-plane. For closed form sloshing modes, this requires all poles to be associated with a common eigenvalue, which, by elementary linear algebra, is shown to be impossible in some standard cases and for other angles that are rational multiples of p.

Deaño, Alfredo

Special function solutions of Painlevé II and IV: asymptotic and numerical study

Special function solutions of the Painlevé equations are important in the theory of orthogonal polynomials, integrable systems and in the study of random matrices. These solutions of Painlevé equations show remarkable structure, since they can be constructed as Wronskian determinants involving a certain seed function, which is a combination of Airy or Weber functions for PII and PIV respectively. In this talk, we will address some asymptotic properties of these functions in the complex plane, as well as their pole fields and some computational challenges.

Gilson, Claire

Constructing and deconstructing solutions in ultra discrete integrable systems

The ultradiscrete KdV equation is a discrete version of the KdV equation with two independent discrete variables n and t. In the early work on the system, the dependent variable was also discrete and took two possible values, 0 or 1. In later works, Hirota and Willox amongst others, generalised the system by allowing the values of the dependent variable to be integers, rationals or real numbers.

In this talk we are going to consider the case where the dependent variable takes real values and show how simple techniques can be used for adding and removing solitons from a solution. This will enable us to fully solve as an initial value problem in a direct manor.

Gómez-Ullate Oteiza, David

Durfee rectangles, exceptional Hermite polynomials and rational solutions to Painlevé equations

We will introduce certain symmetries of Wronskian determinants whose entries are Hermite polynomials, which have a nice combinatorial interpretation in terms of Maya diagrams and Durfee rectangles. Some applications to exceptional Hermite polynomials and rational solutions to Painlevé equations will be also discussed.

Grava, Tamara

Universality of critical behaviour in Hamiltonian PDEs

I study and classify various type of critical behaviours to solutions of Hamiltonian Partial Differential Equations (PDEs) in one and two spatial dimensions that have their origin in optics and fluid dynamic. Painlevé equations and special functions play an important role in the description of these critical behaviours.

Halburd, Rod

Integrable delay-differential equations

Several delay-differential equations with limits to the usual (differential) Painlevé equations will be obtained

a) as the reductions of integrable differential-difference equations,
b) from Bäcklund transformations of Painlevé equations, and
c) by looking for delay-differential equations admitting meromorphic solutions with regular value distribution in the complex plane.

Delay-differential analogues of special cases of the Quispel-Roberts-Thompson (QRT) mapping will also be described.

Himonas, Alex

The unified transform method and well-posedness of nonlinear dispersive equations

The unified transform method (UTM), which is also known as the Fokas transform method, was introduced in late nineties as the analogue of the inverse scattering transform machinery for integrable nonlinear equations on the half-line. It was later understood that it also has significant implications for linear initial-boundary value problems. In this talk, this method is employed in a new direction, namely for showing well-posedness of nonlinear dispersive equations, including the nonlinear Schrödinger equation and the Korteweg-de Vries equation, on the half-line with data in Sobolev spaces.

Hitzazis, Iasonas

Linear elliptic PDEs in a cylindrical domain with a polygonal cross-section

Integral representations for the solution of the Laplace, modified Helmholtz, and Helmholtz equations can be obtained using Green's theorem. However, these representations involve both the Dirichlet and the Neumann values on the boundary, and for a well-posed boundary value problem (BVP) one of these functions is unknown. A new transform method for solving BVPs for linear and for integrable nonlinear partial differential equations (PDEs), usually referred to as the unified transform or the Fokas method, was introduced in the late nineties. For linear elliptic PDEs in two dimensions, this method first, by employing two algebraic equations formulated in the Fourier plane, provides an elegant approach for determining the Dirichlet to Neumann map, i.e., for constructing the unknown boundary values in terms of the given boundary data. Second, this method constructs novel integral representations of the solution in terms of integrals formulated in the complex Fourier plane. In the present paper, we extend this novel approach to the case of the Laplace, modified Helmholtz, and Helmholtz equations, formulated in a three-dimensional cylindrical domain with a polygonal cross-section. This is joint work with A. S. Fokas.

Howls, Chris

Invisible catastrophes: when to turn an asymptotic blind eye

The talk deals with a class of functions that are present across asymptotic modelling that possess subtle asymptotic properties resulting from the coalescence of critical points with singularities. Far from leading to caustic divergences in the integral at these coalescences, the integral is actually asymptotically well-behaved and usual techniques lead to a good numerical approximation. However, delving deeper into the asymptotics uncovers additional caustic complications that, both analytically and numerically would have been better left alone. Examples from acoustics and optics will be presented.

Johnson, Edward
Rotating vortical outflows
A simple, fully-nonlinear, dispersive, quasigeostrophic model is put forward to isolate the vorticity dynamics of coastal outflows. The model is sufficiently simple so as to allow highly accurate, numerical integration of the full problem and also explicit, fully-nonlinear solutions for the evolution of a uniform velocity outflow in the hydraulic limit. The flow evolution depends strongly on the sign of the vorticity of the expelled fluid and on the ratio of the internal Rossby radius to the vortex-source scale, $|V_0 / D^2 \Pi_0|^{\frac{1}{2}}$, of the flow, where D measures the outflow depth, Π_0, the perturbation potential vorticity in the outflow and V_0 the volume flux of the outflow. Comparison of the explicit hydraulic solutions with the numerical integrations shows that the analytical solutions predict the flow development well with differences ascribable to dispersive Rossby waves on the current boundary and changes in the source region captured by the full equations but not present in the hydraulic solutions.

Kalimeris, Kostas
A non-local formulation for two-dimensional water waves
In this talk, we analyse inviscid, irrotational, two dimensional water waves in a bounded domain, using a powerful methodology for studying boundary value problems, provided by the unified transform method, also known as the Fokas method. After discussing this methodology in a more general frame, we will focus on the generation of waves by a moving bottom, as it occurs in tsunamis.

We show that this problem is characterised by two equations which involve only first order derivatives. Under the assumption of "small amplitude" waves, these equations yield a new generalisation of the Boussinesq system which is valid without the long wave approximation. If time allows, the possibility of treating the above problem numerically via a recently introduced numerical technique will also be discussed.

King, John
Complex-plane analysis of diffusive travelling waves
Aspects of the complex-plane properties of travelling-wave propagation in reaction diffusion will be outlined. Situations in which the former provide important insight into the real-line behaviour will be highlighted, alongside respects in which the latter raises challenges for complex analysis.

Kisil, Anastasia
Approximate matrix Wiener-Hopf factorisations and applications to problems in acoustics
First, I will introduce the Wiener-Hopf method which extends the separation of variables technique (in Cartesian coordinate) used to investigate PDEs. It provided analytic and systematic methodology for previously unapproachable problems. One of the problems discussed will be scattering of a sound wave by an infinite periodic grating composed of rigid plates (joint work with I. D. Abrahams).

I will also talk about a matrix Wiener-Hopf problems which is motivated by studying the effect of a finite elastic trailing edge on noise production (joint work with N. Peake, L. Ayton). The approximate factorisation of this matrix with exponential phase factors is achieved using an iterative procedure which makes use of the scalar Wiener-Hopf problem arising for each junction.

Llewellyn Smith, Stefan
Solving matrix Wiener-Hopf problems numerically via Riemann-Hilbert problems
A Riemann-Hilbert (RH) problem asks for the construction of a function that is analytic everywhere in the complex plane except along a given curve where it has a prescribed jump. The Wiener-Hopf (WH) method was developed to solve mixed boundary-value problem; its fundamental essence consists in factoring a function into parts that are analytic in different domains. Explicit formulas to compute this factorization for the scalar case, but no general method is known for matrix problems. The relation between RH problems and the WH method has been known for a long time. Numerical methods to solve RH problems have been the subject of considerable recent work. We examine their use in solving Wiener-Hopf problems, in particular matrix problems. We examine a number of model problems and focus on obtaining numerical results that can be used to compute quantities such as

the far-field amplitude and other properties of the physical solution. It turns out that particular attention needs to be paid to the decay properties of the relevant functions.

Lombardo, Sara

Linear stability analysis of integrable partial differential equations

Analytical methods of the theory of integrable partial differential equations (PDEs) in $1+1$ dimensions have been successfully applied to investigate a number of wave propagation models of physical interest. This talk shows how to address the issue of linear stability of wave solutions by means of these methods. By imitating the standard steps followed when dealing with non integrable equations, we show how the linear stability of solutions of integrable PDEs can be effectively analysed by using their Lax representation. The most relevant application of this scheme is the analysis of the background continuous wave solution. The talk is based on work done in collaboration with Antonio Degasperis, University of Rome La Sapienza, Rome, Italy and Matteo Sommacal, Northumbria University, Newcastle upon Tyne, UK.

Louca, Elena

A new transform approach to biharmonic boundary value problems in polygonal and circular domains

Motivated by modelling challenges arising in microfluidics and low-Reynolds-number swimming, we present a new transform approach for solving biharmonic boundary value problems in two-dimensional polygonal and circular domains. The method is an extension of earlier work by Crowdy & Fokas [Proc. Roy. Soc. A, 460, (2004)] and provides a unified general approach to finding quasi-analytical solutions to a wide range of problems in low-Reynolds-number hydrodynamics and plane elasticity.

Loureiro, Ana

On star-symmetric polynomials with a classical behaviour

I will discuss sequences of polynomials of a single variable that are orthogonal with respect to a vector of weights defined in the complex plane. Such polynomial sequences satisfy a recurrence relation of finite (and fixed) order higher than 2. They share a number of properties that mimic those of standard orthogonality on L_2 spaces. The main focus will be on polynomial sequences possessing a three-star symmetry and whose multiple orthogonality is preserved under the action of the derivative operator.

Mansfield, Elizabeth

Discrete moving frames and discrete variational problems

I will discuss results for Noether's theorem for finite difference variational problems, in terms of a discrete moving frame. The main example will be to derive a difference approximation for Euler's elastica which is symplectic and has all three conservation laws embedded a priori. The result of the numerical experiment will be shown. The approximate Lagrangian is derived by matching, in a sense I will describe, the smooth and the discrete moving frames; in both the smooth and difference cases, it is the frames which provide the key to analysing the Euler Lagrange equations and the conservation laws for Lie group invariant Lagrangians. (Joint work with Ana Rojo-Echeburua, Linyu Peng and Peter Hydon).

Mineev-Weinstein, Mark

Thermodynamics of the Laplacian growth

The methods of equilibrium statistical thermodynamics are applied to Laplacian growth by using its remarkable connection with a random matrix theory. The Laplacian growth equation is obtained from the variation principle and describes adiabatic (quasi-static) thermodynamic processes in the two-dimensional Dyson gas. By using Einstein's theory of thermodynamic fluctuations we consider transitional probabilities between thermodynamic states, which are in a one-to-one correspondence with planar domains. Transitions between these domains are described by the stochastic Laplacian growth equation, while the transitional probabilities coincide with the free-particle propagator on the infinite dimensional complex manifold with the Kahler metric.

Nelson, Rhodri

Outer boundary effects in a petroleum reservoir

We apply an improved calculus based on conformal mapping to model the sweep pattern of waterfloods in bounded reservoirs. Solutions for streamlines and flood advancement obtained with the new calculus method are validated using an independent but more intricate numerical streamline simulation method. Subsequently, the use of the benchmarked calculus is demonstrated in a review of the flow patterns in the Quitman field.

Nemes, Ggerő

Computable error bounds for asymptotic expansions of integrals via resurgence

Resurgence, emerging from the work of Dingle and Écalle, is now a fundamental tool in asymptotic analysis. Resurgence has been used successfully in the asymptotic theory of integrals, difference- and differential equations to obtain exponentially accurate approximations and to understand the Stokes phenomenon. Berry and Howls obtained convenient integral representations for the remainder terms of asymptotic expansions arising from an application of the method of steepest descents. These integral representations were then used by Boyd to obtain error bounds for such asymptotic expansions, in particular for the well-known Stirling asymptotic expansion of the gamma function. In this talk, I will give an alternative representation for the remainders involving the so-called terminant functions. These terminant functions were previously used for exponentially improved expansions and for the computation of the Stokes multipliers of differential equations, but, as I will show, they have a significant role in bounding error terms. It turns out that in many cases, the remainder can be bounded by the absolute value of the first omitted term of the expansion. I will provide applications for special functions such as the incomplete gamma function, the Airy functions and the Bessel functions (both large argument and/or large order). I will briefly compare the results with those arising from differential equation methods by Olver, and show that the new approach often gives much simpler and sharper bounds.

Olde Daalhuis, Adri

Computation of the coefficients appearing in the uniform asymptotic expansions of integrals

The coefficients that appear in uniform asymptotic expansions for integrals are typically very complicated. In the existing literature the majority of the work only give the first two coefficients. In a limited number of papers where more coefficients are given the evaluation of the coefficients near the coalescence points is normally highly numerically unstable. In this paper, we illustrate how well-known Cauchy type integral representations can be used to compute the coefficients in a very stable and efficient manner. We discuss the cases: (i) two coalescing saddles, (ii) two saddles coalesce with two branch points, (iii) a saddle point near an endpoint of the interval of integration. As a special case of (ii) we give a new uniform asymptotic expansion for Jacobi polynomials $P_n^{\alpha,\beta}(z)$ in terms of Laguerre polynomials $L_n^{\alpha}(x)$ as $n \to \infty$ that holds uniformly for z near 1. Several numerical illustrations are included.

Olver, Peter

Dispersive quantization of linear and nonlinear waves

The evolution, through spatially periodic linear dispersion, of rough initial data leads to surprising quantized structures at rational times, and fractal, non-differentiable profiles at irrational times. Similar phenomena have been observed in optics and quantum mechanics, and lead to intriguing connections with exponential sums arising in number theory. Ramifications and recent progress on the analysis, numerics, and extensions to nonlinear wave models will be discussed.

Protas, Bartosz

On the stability of free-boundary problems: a case study in vortex dynamics

Many problems in science and engineering are described in terms of equilibrium shapes on which certain conditions are imposed and which separate regions where the solution may have different properties. A prototypical problem of this type involves 2D inviscid vortex equilibria where constant-vorticity vortex patches are embedded in a potential flow. Methods of complex analysis offer a particularly elegant and efficient description of such systems. Studying linear stability of such free-boundary problems is however challenging as it requires characterization of the sensitivity of the equilibrium solutions with respect to suitable perturbations of the boundary. We will demonstrate that such questions can be in fact systematically addressed using techniques of "shape calculus" applied to formulations based on singular integral equations. In the context of vortex dynamics we use this approach to obtain an equation characterizing the stability of general equilibrium solutions involving vortex patches. Certain classical results of vortex stability are then derived as special cases. Finally, this approach is employed to solve an open problem concerning the linear stability of Hill's vortex to 3D axisymmetric perturbations, which leads to some unexpected findings.

Sheils, Natalie
The heat equation with imperfect thermal contact in a composite medium
The problem of heat conduction in one-dimensional piecewise homogeneous composite materials is examined by providing an explicit solution of the one-dimensional heat equation in each domain. The location of the interfaces is known, but neither temperature nor heat flux are prescribed there. We find a solution using the Unified Transform Method, due to Fokas and collaborators, applied to interface problems and compute solutions numerically.

Smith, David
Nonlocal problems for linear evolution equations
Linear evolution equations, such as the heat equation, are commonly studied on finite spatial domains via initial-boundary value problems. In place of the boundary conditions, we consider "multipoint conditions", where one specifies some linear combination of the solution and its derivative evaluated at internal points of the spatial domain, and "nonlocal" specification of the integral over space of the solution against some continuous weight.

Trichtchenko, Olga
Solutions and stability for flexural-gravity waves
We will describe the methods used to compute solutions to waves under a sheet of ice and show the stability results in different asymptotic regions for these solutions.