A new transform approach to biharmonic boundary value problems in polygonal and circular domains

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Joint work with D. Crowdy (Imperial)

ICMS ACCA Workshop
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The biharmonic equation in polygonal and circular domains

$$\boxed{\nabla^4 \psi(x, y) = 0}$$

- Plane elasticity: ψ is the Airy stress function
- Stokes flows: ψ is the associated streamfunction

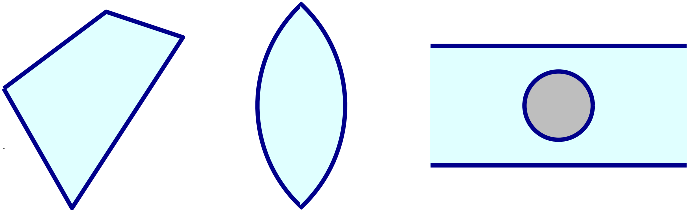

Figure: Examples of polygonal and circular domains

Outline of talk

- The biharmonic equation in Stokes flows
- The unified transform method (Fokas method)

- Part I: Polygonal domains
  - The unified transform method for convex polygons
  - Problem 1: Periodic Stokes flow in a channel

- Part II: Circular domains
  - The unified transform method for circular domains
  - Problem 2: Flow past a circular ridge
  - Problems 3-4: Cylinders near no-slip boundaries

The biharmonic equation in Stokes flows

$$\nabla p = \mu \nabla^2 \mathbf{u}, \quad \nabla . \mathbf{u} = 0$$

where $\mathbf{u} = (u, v)$ is the two dimensional velocity field, p is the fluid pressure and μ is the viscosity.

Since the flow is incompressible, introduce a streamfunction ψ:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Taking the curl of the Stokes equation, it can be shown that

$$\boxed{\nabla^4 \psi = 0} \quad \Rightarrow \quad \frac{\partial^4 \psi}{\partial z^2 \partial \bar{z}^2} = 0$$

General solution of the biharmonic equation:

$$\boxed{\psi(x, y) = \mathrm{Im}[\bar{z} f(z) + g(z)]}$$

The biharmonic equation in Stokes flows

General solution of the biharmonic equation:

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where $f(z)$ and $g(z)$ are analytic functions in the flow region, but can have isolated singularities to model flows of interest.

General solution of the biharmonic equation:

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where $f(z)$ and $g(z)$ are analytic functions in the flow region, but can have isolated singularities to model flows of interest.

Physical quantities of interest can be expressed in terms of $f(z)$ and $g(z)$:

Velocity

$$u - \mathrm{i}v = -\overline{f(z)} + \bar{z} f'(z) + g'(z)$$

Pressure & vorticity

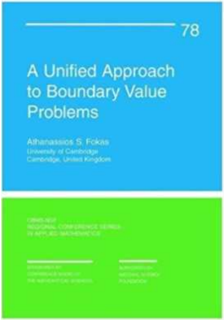
$$\frac{p}{\mu} - \mathrm{i}\omega = 4f'(z)$$

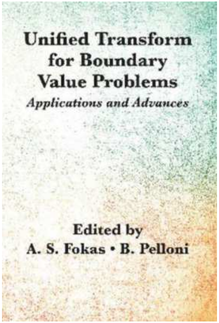
Fluid stress on a boundary component

$$-pn_i + 2\mu e_{ij} n_j = 2\mu\mathrm{i}\frac{dH}{ds}, \quad \text{where} \quad H(z,\bar{z}) = f(z) + z\overline{f'(z)} + \overline{g'(z)}$$

Fokas A.S. (1997), A unified transform method for solving linear and certain nonlinear PDEs.

- Generalization of the Fourier transform method $\rightarrow$ involves simultaneous spectral analysis in both x and y.





The unified transform method (Fokas method)

Laplace's equation:

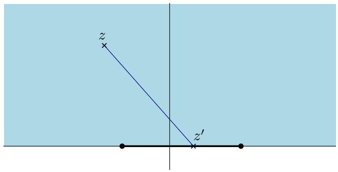
- Fokas & Kapaev (2003) presented a novel constructive method for the solution of mixed boundary value problems for Laplace's equation in a convex polygon $\rightarrow$ generalized Fourier transforms for convex polygons.
- Crowdy (2015) has recently presented an extension to circular domains of the results derived by Fokas & Kapaev (2003). The newly derived transforms are generalizations of the classical Fourier and Mellin transforms to general circular domains.

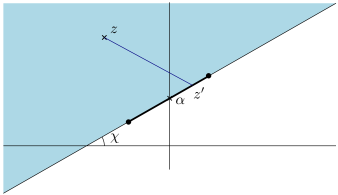
Biharmonic equation:

- A transform method applicable to boundary value problems for biharmonic fields in simply connected polygonal domains was developed by Crowdy & Fokas (2004), with further developments made more recently by Dimakos & Fokas (2015).
- In this talk, we show how to extent the transform method for circular domains for biharmonic boundary value problems.

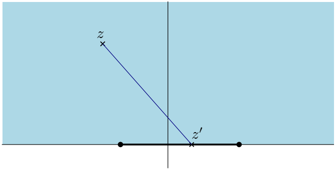
Part I
Polygonal domains

The unified transform method for convex polygons



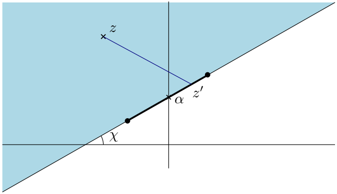


Crowdy (2015)

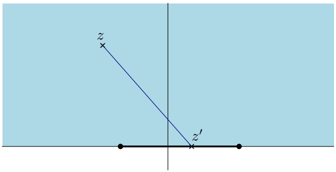


$$0 < \arg[z - z'] < \pi$$

$$\int_0^\infty e^{ik(z-z')}dk = \left[\frac{e^{ik(z-z')}}{\mathrm{i}(z'-z)}\right]_0^\infty = \frac{1}{\mathrm{i}(z'-z)}$$

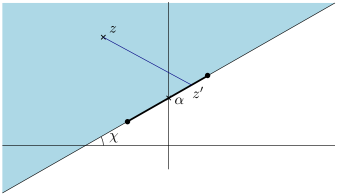


Crowdy (2015)



$$0 < \arg[z - z'] < \pi$$

$$\int_0^\infty e^{\mathrm{i}k(z-z')} dk = \left[\frac{e^{\mathrm{i}k(z-z')}}{\mathrm{i}(z'-z)} \right]_0^\infty = \frac{1}{\mathrm{i}(z'-z)}$$

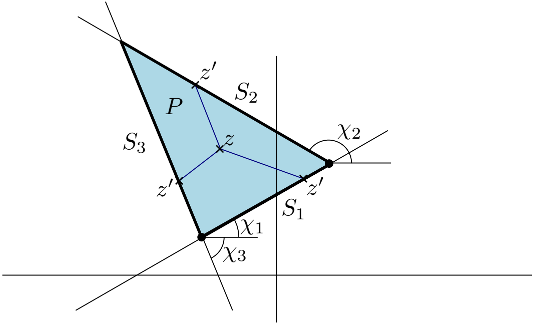


$$0 < \arg[e^{-\mathrm{i}\chi}(z-\alpha) - e^{-\mathrm{i}\chi}(z'-\alpha)] < \pi$$

$$\frac{1}{z'-z} = \mathrm{i} \int_0^\infty e^{\mathrm{i}e^{-\mathrm{i}\chi}k(z-z')} e^{-\mathrm{i}\chi} dk$$

Crowdy (2015)

The unified transform method for convex polygons



For a function $h(z)$ analytic in domain P, Cauchy's integral formula provides that for $z \in P$:

$$h(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial P} \frac{h(z')dz'}{z' - z} = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \int_{S_j} h(z') \left[\frac{1}{z' - z} \right] dz'$$

Replace the Cauchy kernel for each side by

$$\frac{1}{z' - z} = \mathrm{i}\int_0^{\infty} e^{\mathrm{i}e^{-\mathrm{i}\chi_j}k(z-z')}e^{-\mathrm{i}\chi_j}\,dk$$

The unified transform method for convex polygons

Replace the Cauchy kernel for each side by

$$\frac{1}{z'-z} = \mathrm{i}\int_0^\infty e^{\mathrm{i}e^{-\mathrm{i}\chi_j}k(z-z')}e^{-\mathrm{i}\chi_j}\,dk$$

to find the following **transform pair**:

$$h(z) = \frac{1}{2\pi}\sum_{j=1}^{N}\int_0^\infty \rho_{jj}(k)e^{-\mathrm{i}\chi_j}e^{\mathrm{i}e^{-\mathrm{i}\chi_j}kz}\,dk$$

$$\rho_{jj}(k) = \int_{S_j} h(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_j}kz'}\,dz'$$

For integers m, n between 1 and N, we define the **spectral matrix**

$$\rho_{mn}(k) = \int_{S_n} h(z')e^{-\mathrm{i}e^{-\mathrm{i}\chi_m}kz'}dz'$$

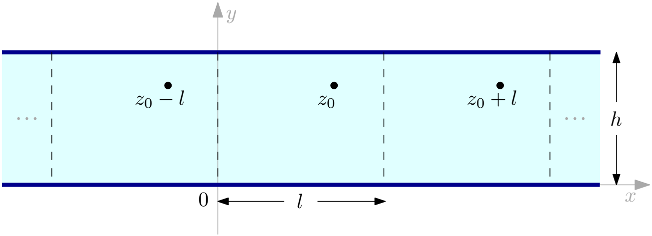
which are called **spectral functions**.

Global relations:

For any $k \in \mathbb{C}$ and for any $m = 1, ..., N$,

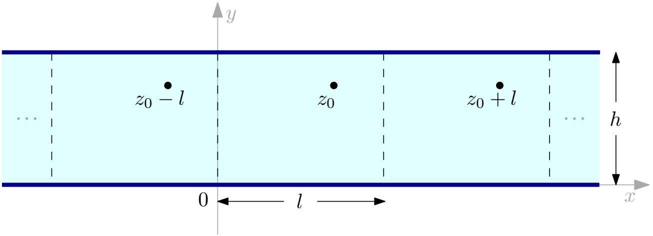
$$\sum_{j=1}^{N} \rho_{mn}(k) = 0$$

Problem 1: Periodic Stokes flow in a channel

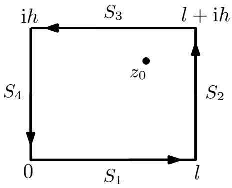


Periodic array of point singularities in a channel

Problem 1: Periodic Stokes flow in a channel



Periodic array of point singularities in a channel



$$f(z) = f_s(z) + \hat{f}(z)$$
$$g'(z) = g_s'(z) + \hat{g}'(z)$$

where $f_s(z)$, $g_s'(z)$ are known functions (associated to the point singularity at z_0) and $\hat{f}(z)$, $\hat{g}'(z)$ are unknown analytic functions.

[Pozrikidis (1992)]

Step 1 - Function representations

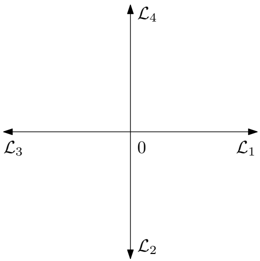
$$f(z) = f_s(z) + \hat{f}(z)$$
$$g'(z) = g_s'(z) + \hat{g}'(z)$$

$$\hat{f}(z) = \frac{1}{2\pi}\left[\sum_{j=1}^{4}\int_{\mathcal{L}_j}\rho_j(k)e^{\mathrm{i}kz}dk\right]$$

$$\rho_j(k) = \int_{S_j}\hat{f}(z)e^{-\mathrm{i}kz}dz$$

Global relation:

$$\sum_{j=1}^{4}\rho_j(k) = 0, \quad \text{for } k \in \mathbb{C}$$

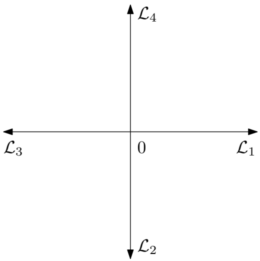


Step 1 - Function representations

$$f(z) = f_s(z) + \hat{f}(z)$$
$$g'(z) = g_s'(z) + \hat{g}'(z)$$

$$\hat{g}'(z) = \frac{1}{2\pi}\left[\sum_{j=1}^{4}\int_{\mathcal{L}_j}\hat{\rho}_j(k)e^{\mathrm{i}kz}dk\right]$$

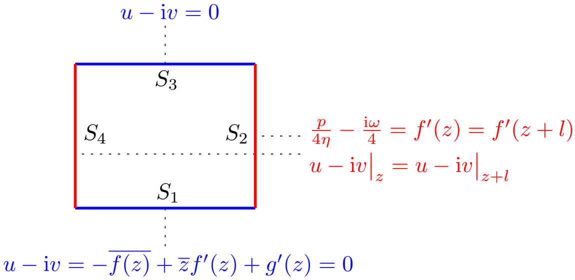
$$\hat{\rho}_j(k) = \int_{S_j}\hat{g}'(z)e^{-\mathrm{i}kz}dz$$



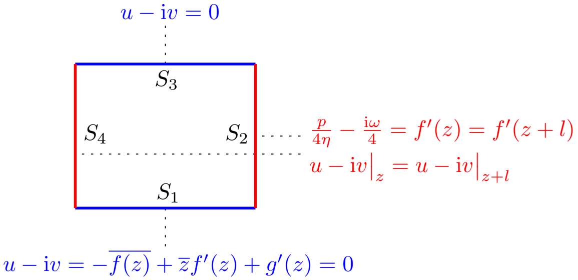
Global relation:

$$\sum_{j=1}^{4}\hat{\rho}_j(k) = 0, \quad \text{for } k \in \mathbb{C}$$

Step 2 - Spectral analysis



Step 2 - Spectral analysis



Side S_1: The no-slip boundary condition can be written as

$$u - \mathrm{i}v = 0 \;\Rightarrow\; -\overline{\hat{f}(z)} + z\hat{f}'(z) + \hat{g}'(z) = \overline{f_s(z)} - zf_s'(z) - g_s'(z).$$

We multiply this by $e^{-\mathrm{i}kz}$ and integrate along the lower boundary:

$$-\int_0^l \overline{\hat{f}(z)} e^{-\mathrm{i}kz} dz + \int_0^l z\hat{f}'(z) e^{-\mathrm{i}kz} dz + \int_0^l \hat{g}'(z) e^{-\mathrm{i}kz} dz = R_1(k).$$

The previous expression can be written as

$$-\overline{\rho_1}(-k) - \frac{\partial[k\rho_1(k)]}{\partial k} + \hat{\rho}_1(k) + a(k) = R_1(k).$$

Similarly, we can find relations between the other spectral functions:

$$\begin{aligned}
-e^{2kh}\overline{\rho_3}(-k) - \frac{\partial[k\rho_3(k)]}{\partial k} + 2kh\rho_3(k) + \hat{\rho}_3(k) + c(k) &= R_3(k), \\
\rho_4(k) + e^{\mathrm{i}kl}\rho_2(k) + b(k) &= R_2(k), \\
-\mathrm{i}kl\rho_4(k) + \hat{\rho}_4(k) + e^{\mathrm{i}kl}\hat{\rho}_2(k) + d(k) &= R_4(k).
\end{aligned}$$

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$$-\overline{\rho_1}(-k) - \frac{\partial[k\rho_1(k)]}{\partial k} + \hat{\rho}_1(k) + a(k) = R_1(k).$$

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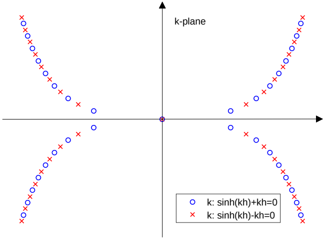
$$\begin{aligned} -e^{2kh}\overline{\rho_3}(-k) - \frac{\partial[k\rho_3(k)]}{\partial k} + 2kh\rho_3(k) + \hat{\rho}_3(k) + c(k) &= R_3(k), \\ \rho_4(k) + e^{\mathrm{i}kl}\rho_2(k) + b(k) &= R_2(k), \\ -\mathrm{i}kl\rho_4(k) + \hat{\rho}_4(k) + e^{\mathrm{i}kl}\hat{\rho}_2(k) + d(k) &= R_4(k). \end{aligned}$$

On use of the global relations, we find that

$$\rho_1(k) = \frac{2khW(k) - (e^{2kh} - 1)\overline{W}(-k)}{\Delta(k)}, \qquad \Delta(k) \equiv 4[\sinh^2(kh) - k^2h^2],$$

where $W(k)$ is in terms of $\{\rho_4(k), \hat{\rho}_4(k)\}$, $\{\hat{f}(0), \hat{f}(\mathrm{i}h)\}$ and known quantities.

Step 3 - Solution scheme



$\rho_1(k)$ is analytic everywhere in $\mathbb{C}$

$\Rightarrow$ its numerator* must vanish at zeros of $\Delta(k)$ in $\mathbb{C}$.

Step 3 - Solution scheme

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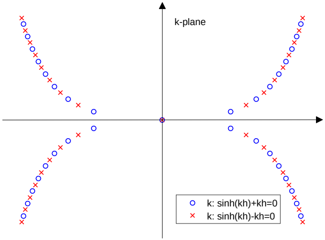


Represent unknown boundary data on side S_4 using Chebyshev basis:

$$\hat{f}(z) = \sum_{m=0}^{M} a_m T_m(s), \qquad \hat{g}(z) = \sum_{m=0}^{M} b_m T_m(s),$$

where $T_m(s)$ are the Chebyshev polynomials. We have:

$$\rho_4(k) = \sum_{m=0}^{M} a_m T(k,m), \qquad \hat{\rho}_4(k) = \sum_{m=0}^{M} b_m U(k,m).$$

*numerator in terms of $\{\rho_4(k), \hat{\rho}_4(k)\}$, $\{\hat{f}(0), \hat{f}(\mathrm{i}h)\}$ and known quantities.

Problem 1: Periodic Stokes flow in a channel

- Formulate an overdetermined linear system for the unknown coefficients $\{a_m, b_m | m = 0, ..., M\}$.
- Once the coefficients are computed, all the unknown spectral functions can be found; these can be used to compute $\hat{f}(z)$, $\hat{g}'(z)$.
- For only $M = 10$, the maximum relative error is $10^{-7}\%$ compared to an alternative fast method using function theory.

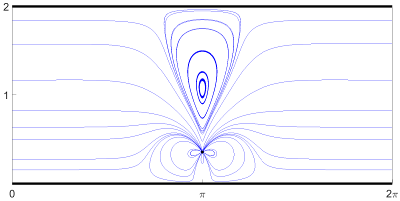
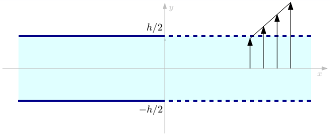


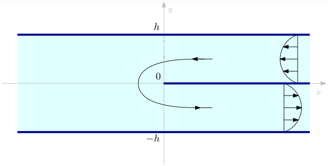
Figure: Streamline pattern for a periodic stresslet in a channel

D. Crowdy & E. Luca, Periodic arrays of singularities in two-dimensional Stokes flow (in preparation).

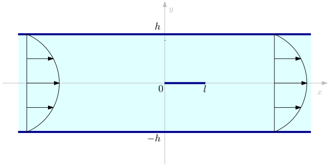
Other problems in polygonal domains



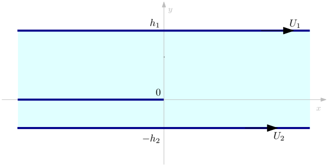
(a) Luchini, Manzo & Pozzi (1991), Jeong (2001)



(b) Jeong (2001)



(c) Kim & Chung (1984)



(d) Abrahams, Davis & Llewellyn Smith (2008)

- Impose continuity conditions across common ‘boundaries’.

D. Crowdy & E. Luca, Solving Wiener-Hopf problems without kernel factorization, (2014).

Part II
Circular domains

ICMS 2012

International Centre for Mathematical Sciences

'Mathematics knows no races or geographic boundaries; for mathematics, the cultural worl

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Boundary value problems for linear elliptic and integrable PDEs: theory and computation

May 28, 2012 - Jun 01, 2012

ICMS 15 South College Street

Organisers

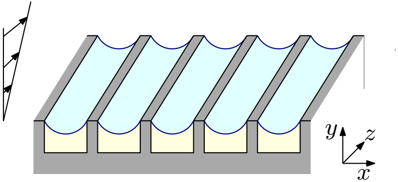
Name	Institution
Bona, Jerry	University of Illinois at Chicago
Chandler-Wilde, Simon	University of Reading
Fornberg, Bengt	University of Colorado
Pelloni, Beatrice	Heriot-Watt University

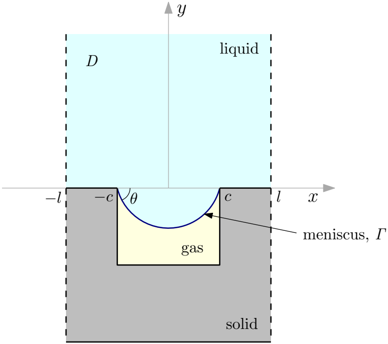
The aim of this workshop is to bring together researchers who have contributed to recent developments in the theory and computation of boundary value problems for certain important partial differential equations, from varying perspectives and using disparate methods.

The meeting will focus on the following themes:

- boundary value problems for nonlinear PDEs, integrablibility formalism and PDE techniques
- boundary value problems for linear elliptic PDEs, Wiener-Hopf methods and high frequency scattering
- new numerical formalisms and comparison with existing numerical techniques, with an emphasis on high frequency wave problems.

Application: Superhydrophobic surfaces





Mixed boundary value problem

- for a harmonic field, if the flow is in the longitudinal direction
- for a biharmonic field, if the flow is in the transverse direction

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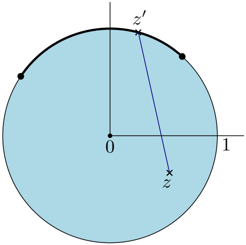
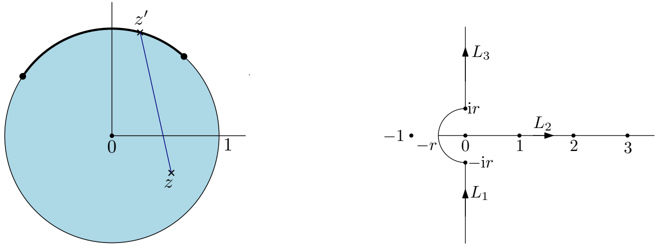
E. Luca, J.S. Marshall & G. Karamanis, Longitudinal shear flow over a bubble mattress with curved menisci: arbitrary protrusion angle and solid fraction, (submitted).

The unified transform method for circular domains





For a function $h(z)$ analytic in the interior P of the unit disc, Cauchy's integral formula provides that for $z \in P$:

$$h(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|z'|=1} \frac{h(z')dz'}{z' - z}$$

The Cauchy kernel has the spectral representation

$$\frac{1}{z' - z} = \int_{L_1} \frac{1}{1 - e^{2\pi \mathrm{i} k}} \frac{z^k}{z'^{k+1}} dk + \int_{L_2} \frac{z^k}{z'^{k+1}} dk + \int_{L_3} \frac{e^{2\pi \mathrm{i} k}}{1 - e^{2\pi \mathrm{i} k}} \frac{z^k}{z'^{k+1}} dk$$

Crowdy (2015)

The unified transform method for circular domains

Replace the Cauchy kernel and find the following **transform pair for the interior of the unit disc**:

$$h(z) = \frac{1}{2\pi \mathrm{i}} \left[\int_{L_1} \frac{\rho(k)}{1 - e^{2\pi \mathrm{i} k}} z^k dk + \int_{L_2} \rho(k) z^k dk + \int_{L_3} \frac{\rho(k) e^{2\pi \mathrm{i} k}}{1 - e^{2\pi \mathrm{i} k}} z^k dk \right]$$

$$\rho(k) = \oint_{|z'|=1} \frac{h(z')}{z'^{k+1}} dz'$$

Replace the Cauchy kernel and find the following **transform pair for the interior of the unit disc**:

$$h(z) = \frac{1}{2\pi \mathrm{i}}\left[\int_{L_1} \frac{\rho(k)}{1-e^{2\pi \mathrm{i}k}} z^k dk + \int_{L_2} \rho(k) z^k dk + \int_{L_3} \frac{\rho(k)e^{2\pi \mathrm{i}k}}{1-e^{2\pi \mathrm{i}k}} z^k dk\right]$$

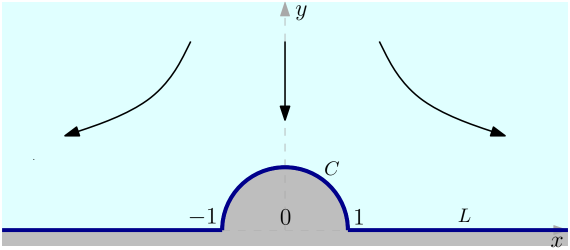
$$\rho(k) = \oint_{|z'|=1} \frac{h(z')}{z'^{k+1}} dz'$$

Global relation:

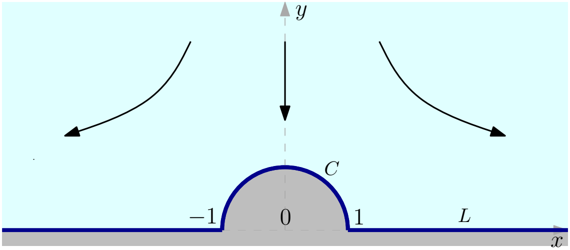
$$\rho(k) = 0, \quad k \in -\mathbb{N}$$

Setting $z \mapsto z'/z$ above, one can obtain the transform pair for the exterior of the unit disc.

Problem 2: Flow past a circular ridge



Stagnation point flow past a semicircular ridge



Stagnation point flow past a semicircular ridge

Step 1 - Function representations

$$f(z) = \frac{i\alpha z^2}{4} + \hat{f}(z)$$

$$g'(z) = -\frac{3i\alpha z^2}{4} + \hat{g}'(z)$$

[Davis & O'Neill (1977)]

$$\hat{f}(z) = \frac{1}{2\pi}\int_0^\infty \rho_{11}(k)e^{\mathrm{i}kz}dk$$
$$-\frac{1}{2\pi\mathrm{i}}\left[\int_{L_1}\frac{\rho_{22}(k)}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk + \int_{L_2}\rho_{22}(k)\frac{1}{z^{k+1}}dk + \int_{L_3}\frac{\rho_{22}(k)e^{2\pi\mathrm{i}k}}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk\right],$$

where

$$\rho_{11}(k) = \int_L \hat{f}(z)e^{-\mathrm{i}kz}dz, \qquad \rho_{22}(k) = -\int_C \hat{f}(z)z^k dz,$$
$$\rho_{12}(k) = -\int_C \hat{f}(z)e^{-\mathrm{i}kz}dz, \qquad \rho_{21}(k) = \int_L \hat{f}(z)z^k dz.$$

Global relations:

$$\rho_{11}(k) + \rho_{12}(k) = 0, \quad k < 0,$$
$$\rho_{21}(k) + \rho_{22}(k) = 0, \quad k \in -\mathbb{N}.$$

The two global relations are equivalent statements of analyticity of $\hat{f}(z)$ in the fluid domain.

$$\hat{g}'(z) = \frac{1}{2\pi}\int_0^\infty \hat{\rho}_{11}(k)e^{\mathrm{i}kz}dk$$
$$-\frac{1}{2\pi\mathrm{i}}\left[\int_{L_1}\frac{\hat{\rho}_{22}(k)}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk + \int_{L_2}\hat{\rho}_{22}(k)\frac{1}{z^{k+1}}dk + \int_{L_3}\frac{\hat{\rho}_{22}(k)e^{2\pi\mathrm{i}k}}{1-e^{2\pi\mathrm{i}k}}\frac{1}{z^{k+1}}dk\right],$$

where

$$\hat{\rho}_{11}(k) = \int_L \hat{g}'(z)e^{-\mathrm{i}kz}dz, \qquad \hat{\rho}_{22}(k) = -\int_C \hat{g}'(z)z^k dz,$$
$$\hat{\rho}_{12}(k) = -\int_C \hat{g}'(z)e^{-\mathrm{i}kz}dz, \qquad \hat{\rho}_{21}(k) = \int_L \hat{g}'(z)z^k dz.$$

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The two global relations are equivalent statements of analyticity of $\hat{g}'(z)$ in the fluid domain.

Problem 2: Flow past a circular ridge

Step 2 - Spectral analysis

- Analyze the spectral relations arising from the boundary conditions,
- Use the global relations
- Identify special points in the spectral plane whereby information on a reduced set of unknown spectral functions can be determined:

$$(k+1)\rho_{22}(k) + \overline{\rho_{22}}(k) - (k-1)\rho_{22}(k-2) + \overline{\rho_{22}}(-k-2) = R(k), \text{ for } k \in -\mathbb{N}.$$

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Step 3 - Solution scheme

- Function representation: represent unknown boundary data using a Chebyshev basis

$$\hat{f}(z(s)) = \sum_{m=0}^{\infty} a_m T_m(s) \quad \Rightarrow \quad \rho_{22}(k) = \sum_{m=0}^{\infty} a_m T(k,m)$$

- Formulate a linear system and solve for the unknown coefficients. Once these are computed, all the unknown spectral functions can be found.

Problem 2: Flow past a circular ridge

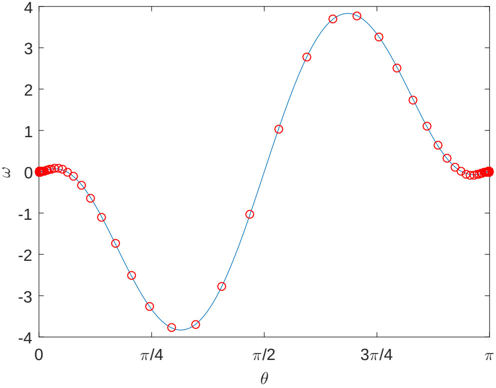
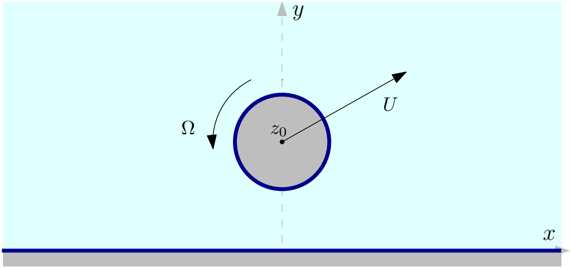


Figure: Vorticity ω along the semicircular ridge $z = e^{\mathrm{i}\theta}$, $\theta = [0, \pi]$ as a function of θ. Comparison of our transform approach (solid line) and solution by Davis & O'Neill (1977) (circles). Our results are for a small truncation parameter ($M = 12$).

Problem 3: Cylinder near a wall



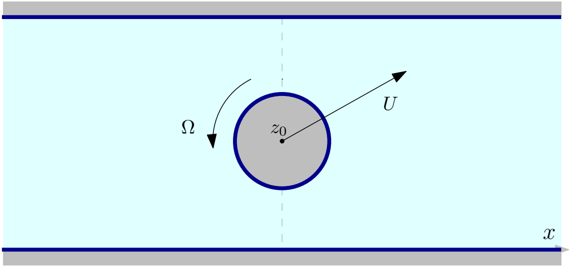
A translating and rotating cylinder with complex speed U and angular velocity Ω

In Stokes flows, forces and torques are linearly related to linear and angular velocities.

Aim: Given U and Ω, compute forces and torque acting on the cylinder, i.e. compute the so called mobility/resistance matrix.

[Jeffrey & Onishi (1981)]

Problem 4: Cylinder in a channel



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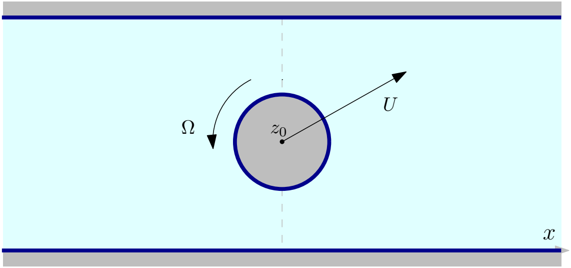
[Dvinsky & Popel (1987), Jeong & Jang (2014), Jeong & Yoon (2014)]

Problem 4: Cylinder in a channel



A translating and rotating cylinder with complex speed U and angular velocity Ω

Step 1 - Function representations

$$f(z) = \lambda \log\left[\tanh\left(\frac{\pi}{2h}(z - z_0)\right)\right] + \hat{f}(z)$$

$$g'(z) = -\overline{\lambda} \log\left[\tanh\left(\frac{\pi}{2h}(z - z_0)\right)\right] + \hat{g}'(z)$$

Problem 4: Cylinder in a channel

$$\hat{f}(z) = \frac{1}{2\pi}\int_0^{\infty} \rho_{11}(k) e^{\mathrm{i}kz} dk + \frac{1}{2\pi}\int_0^{-\infty} \rho_{33}(k) e^{\mathrm{i}kz} dk$$
$$- \frac{1}{2\pi \mathrm{i}}\left[\int_{L_1} \frac{\rho_{22}(k)}{1 - e^{2\pi \mathrm{i} k}} \frac{1}{(z - z_0)^{k+1}} dk + \int_{L_2} \frac{\rho_{22}(k)}{(z - z_0)^{k+1}} dk \right.$$
$$\left. + \int_{L_3} \frac{\rho_{22}(k) e^{2\pi \mathrm{i} k}}{1 - e^{2\pi \mathrm{i} k}} \frac{1}{(z - z_0)^{k+1}} dk \right],$$

$$\rho_{11}(k) = \int_{-\infty}^{\infty} \hat{f}(z) e^{-\mathrm{i}kz} dz, \qquad \rho_{22}(k) = -\oint_{|z - z_0| = 1} \hat{f}(z)(z - z_0)^k dz,$$

$$\rho_{33}(k) = \int_{\infty + \mathrm{i}h}^{-\infty + \mathrm{i}h} \hat{f}(z) e^{-\mathrm{i}kz} dz.$$

$$\rho_{21}(k) = \int_{-\infty}^{\infty} \hat{f}(z)(z - z_0)^k dz, \qquad \rho_{23}(k) = \int_{\infty + \mathrm{i}h}^{-\infty + \mathrm{i}h} \hat{f}(z)(z - z_0)^k dz,$$

$$\rho_{12}(k) = \rho_{32}(k) = -\oint_{|z - z_0| = 1} \hat{f}(z) e^{-\mathrm{i}kz} dz$$

and $\rho_{31}(k) = \rho_{11}(k)$ and $\rho_{13}(k) = \rho_{33}(k)$.

Global relations:

$$\rho_{11}(k) + \rho_{12}(k) + \rho_{13}(k) = 0, \quad k \in \mathbb{R},$$
$$\rho_{31}(k) + \rho_{32}(k) + \rho_{33}(k) = 0, \quad k \in \mathbb{R},$$

which are equivalent, and

$$\rho_{21}(k) + \rho_{22}(k) + \rho_{23}(k) = 0, \quad k \in -\mathbb{N}.$$

(Similar expressions can be written for $\hat{g}'(z)$ and $\hat{\rho}_{ij}(k)$, $i, j = 1, 2, 3$)

Problem 4: Cylinder in a channel

Global relations:

$$\rho_{11}(k)+\rho_{12}(k)+\rho_{13}(k)=0, \quad k\in\mathbb{R},$$
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which are equivalent, and

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(Similar expressions can be written for $\hat{g}'(z)$ and $\hat{\rho}_{ij}(k)$, $i,j=1,2,3$)

Step 2 - Spectral analysis

- Analyze the spectral relations arising from the boundary conditions,
- Use **both** global relations
- Identify special points in the spectral plane whereby information on a reduced set of unknown spectral functions can be determined

$$\oint_{|z-z_0|=1}\hat{f}(z)(z-z_0)^{-n}dz=\int_0^\infty A(k)dk, \quad n\in\mathbb{N},$$
$$\oint_{|z-z_0|=1}\hat{g}'(z)(z-z_0)^{-n}dz=\int_0^\infty B(k)dk, \quad n\in\mathbb{N}.$$

Step 3 - Solution scheme

- Function representation: represent unknown boundary data in terms of a Laurent series

$$\hat{f}(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m$$

- Formulate a linear system and solve for the unknown coefficients. Once these are computed, all the unknown spectral functions can be found.

$$a_{n-1} = \sum_{m=-M}^{M} a_m A_{nm} + \sum_{m=-M}^{M} \overline{a_m} B_{nm} + C_n, \quad n \in \mathbb{N}$$

$$(n-1)a_{n-1} + 2nz_0 a_n - (n+1)a_{n+1} + \overline{a_{-n+1}} = \sum_{m=-M}^{M} a_m A'_{nm} + \sum_{m=-M}^{M} \overline{a_m} B'_{nm} + C'_n, \quad n \in \mathbb{N}$$

Problem 4: Cylinder in a channel

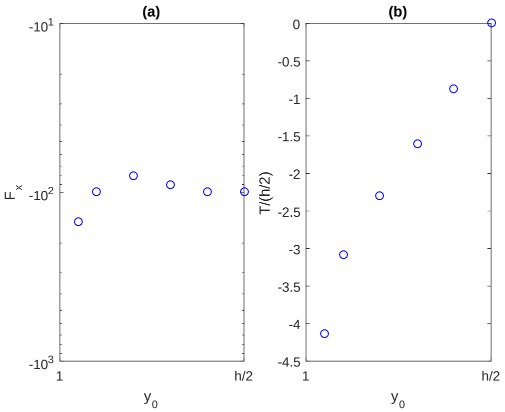

Figure: Motion parallel to the channel walls ($U = 1$). The results shown are, in general, for a small truncation parameter ($M = 10$). However, as cylinder approaches the no-slip wall, M should be increased.

Problem 4: Cylinder in a channel

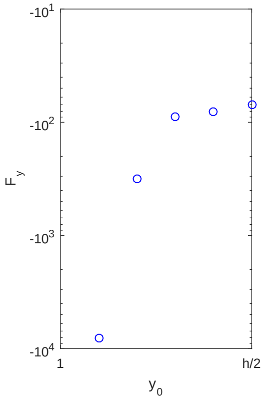

Figure: Motion perpendicular to the channel walls ($U = \mathrm{i}$).

Problem 4: Cylinder in a channel

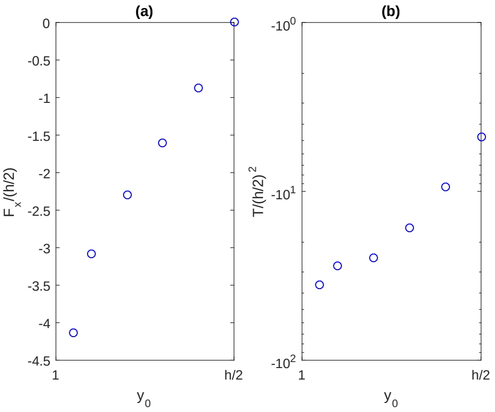

Figure: Pure rotation ($\Omega = 1$).

Summary

- Our transform approach can be applied to a very large class of circular domain geometries in an algorithmic way; all that changes between examples is the analysis of the global relations.
- We have found that in many cases, to obtain high accuracy, only a few coefficients in a Fourier (Laurent) or Chebyshev expansion of a single unknown boundary function need to be determined, with all other spectral functions following by back substitution.
- Our approach might be called "quasi-analytical" in that only a small number of unknown coefficients are needed to represent the global solution even though those coefficients must be found numerically by solving a low-order linear system.

E. Luca & D. Crowdy, A transform method for the biharmonic equation in multiply connected circular domains (submitted).