The Heat Equation with imperfect contact

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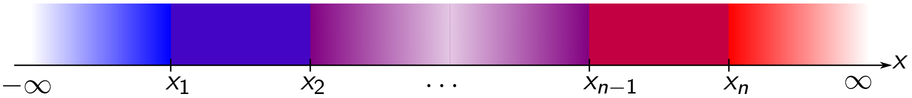
Applied and Computational Complex Analysis

May 8, 2017

Acknowledgements

- International Centre for Mathematical Sciences (ICMS)
- Institute for Math and Its Applications (IMA)
- National Science Foundation (NSF-DGE-0718124)

Interface Problems



Interface Problems

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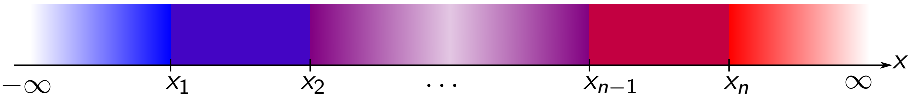

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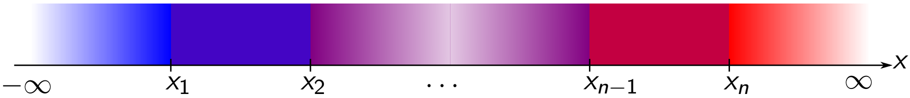

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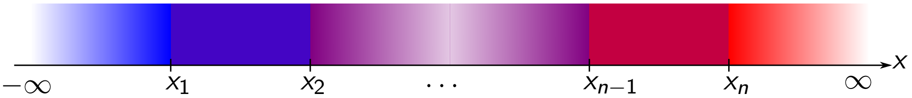

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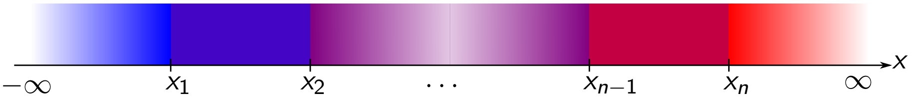
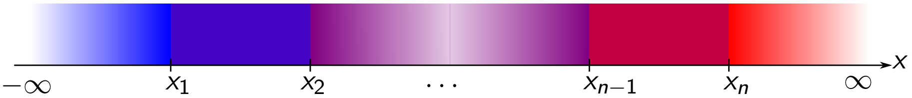


Interface Problems

- The domain and the location of the interface(s) are fixed and known.
- No functions are explicitly known at the interface(s).
- We only prescribe initial conditions.
- In effect, the solution of the PDE in one domain provides the boundary conditions for the solutions of the PDEs in the adjacent domains.
- The Unified Transform Method (UTM) alternatively called the Fokas Method, has been used successfully to solve these problems in the case of many important equations.



The Heat Equation

We consider the heat equation on a finite domain with n interfaces

$$u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j, \quad 1 \leq j \leq n+1,$$

The Heat Equation

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$$u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j, \quad 1 \le j \le n+1,$$

$$u^{(j)}(x,0) = u_0^{(j)}(x), \qquad x_{j-1} < x < x_j, \quad 1 \le j \le n+1,$$

$$\begin{aligned} \beta_1 u^{(1)}(x_0,t) + \beta_2 u_x^{(1)}(x_0,t) &= f_1(t), \qquad t > 0, \\ \beta_3 u^{(n+1)}(x_{n+1},t) + \beta_4 u_x^{(n+1)}(x_{n+1},t) &= f_2(t), \qquad t > 0. \end{aligned}$$

The Heat Equation

If each layer is in perfect thermal contact then the interface conditions are

$$u^{(j)}(x_j, t) = u^{(j+1)}(x_j, t), \qquad t > 0,$$
$$\sigma_j^2 u_x^{(j)}(x_j, t) = \sigma_{j+1}^2 u_x^{(j+1)}(x_j, t), \qquad t > 0.$$

However, if the thermal contact is imperfect we prescribe the interface conditions

$$\sigma_j^2 u_x^{(j)}(x_j, t) = H_j \left(u^{(j+1)}(x_j, t) - u^{(j)}(x_j, t) \right), \qquad t > 0,$$
$$\sigma_{j+1}^2 u_x^{(j+1)}(x_j, t) = H_j \left(u^{(j+1)}(x_j, t) - u^{(j)}(x_j, t) \right), \qquad t > 0,$$

where $H_j \neq 0$ is the contact transfer coefficient at $x = x_j$ and $1 \leq j \leq n$.

1. Local relation

This system of PDEs is equivalent to

$$\left(e^{-ikx+\omega_j(k)t}u^{(j)}\right)_t = \left(e^{-ikx+\omega_j(k)t}\sigma_j^2(u_x^{(j)} + iku^{(j)})\right)_x,$$

with $\omega_j(k) = (\sigma_j k)^2$.

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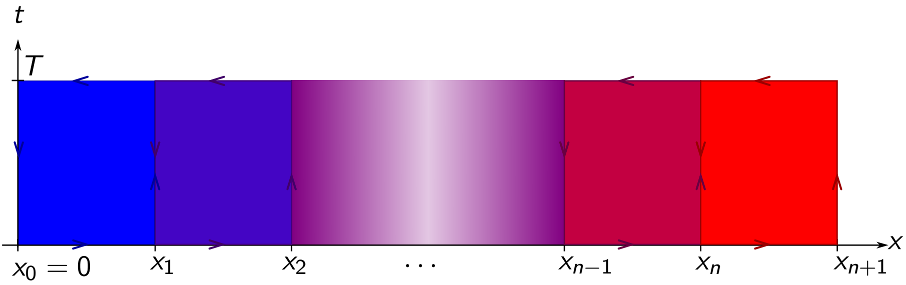
with $\omega_j(k) = (\sigma_j k)^2$.
Integrating over the domain $\mathcal{R}_j = \{(x,t) : x_{j-1} < x < x_j, 0 < t < T\}$

$$\oint_{\mathcal{R}_j} \left(e^{-ikx+\omega_j(k)t}u^{(j)}\right)_t - \left(e^{-ikx+\omega_j(k)t}\sigma_j^2(u_x^{(j)} + iku^{(j)})\right)_x \mathrm{d}x\,\mathrm{d}t = 0,$$

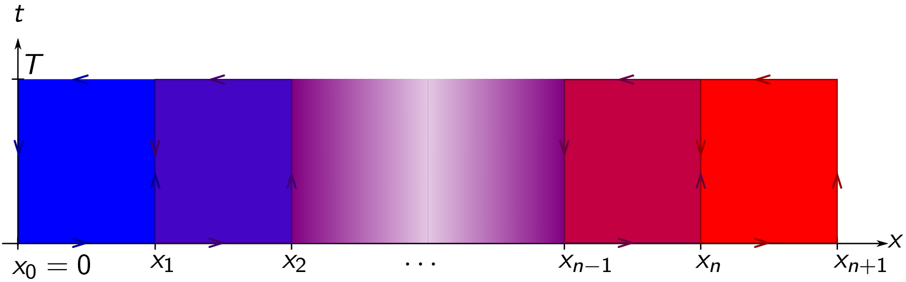
2. Global relation

$$\oint_{\mathcal{R}_j} \left(e^{-ikx+\omega_j(k)t} u^{(j)}\right)_t - \left(e^{-ikx+\omega_j(k)t} \sigma_j^2 (u_x^{(j)} + iku^{(j)})\right)_x \,\mathrm{d}x\,\mathrm{d}t = 0,$$



2. Global relation

$$\int_{x_{j-1}}^{x_j} \int_0^T \left(e^{-ikx+\omega_j(k)t} u^{(j)} \right) \mathrm{d}x + \left(e^{-ikx+\omega_j(k)t} \sigma_j^2 (u_x^{(j)} + iku^{(j)}) \right) \mathrm{d}t = 0,$$



2. Global relation

The global relation relates the function, its values at the interface, and the initial condition.

$$
\begin{aligned}
0 = &\int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{-ikx+\omega_j(k)T} u^{(j)}(x, T)\,\mathrm{d}x \\
&+ \int_0^T \sigma_j^2 e^{-ikx_j+\omega_j(k)s} \left(u_x^{(j)}(x_j, s) + iku^{(j)}(x_j, s)\right) \mathrm{d}s \\
&- \int_0^T \sigma_j^2 e^{-ikx_{j-1}+\omega_j(k)s} \left(u_x^{(j)}(x_{j-1}, s) + iku^{(j)}(x_{j-1}, s)\right) \mathrm{d}s,
\end{aligned}
$$

for $1 \leq j \leq n+1$.

2. Global relation

Define

$$\hat{u}^{(j)}(k,t) = \int_{x_{j-1}}^{x_j} e^{-ikx} u^{(j)}(x,t)\,\mathrm{d}x, \qquad x_{j-1} < x < x_j, \qquad 0 < t < T,$$

$$\hat{u}_0^{(j)}(k) = \int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x, \qquad x_{j-1} < x < x_j,$$

$$g_0^{(j)}(\omega,t) = \int_0^t e^{\omega s} u^{(j)}(x_{j-1},s)\,\mathrm{d}s, \qquad 0 < t < T,$$

$$g_1^{(j)}(\omega,t) = \int_0^t e^{\omega s} u_x^{(j)}(x_{j-1},s)\,\mathrm{d}s \qquad 0 < t < T,$$

$$h_0^{(j)}(\omega,t) = \int_0^t e^{\omega s} u^{(j)}(x_j,s)\,\mathrm{d}s, \qquad 0 < t < T,$$

$$h_1^{(j)}(\omega,t) = \int_0^t e^{\omega s} u_x^{(j)}(x_j,s)\,\mathrm{d}s, \qquad 0 < t < T,$$

All of these integrals are proper integrals defined for $k \in \mathbb{C}$.

2. Global relation

The global relations become

$$e^{\omega_j(k)T}\hat{u}^{(j)}(k,T) = \hat{u}_0^{(j)}(k) + \sigma_j^2 e^{-ikx_j}\left(h_1^{(j)}(\omega_j(k),T) + ikh_0^{(j)}(\omega_j(k),T)\right) - \sigma_j^2 e^{-ikx_{j-1}}\left(g_1^{(j)}(\omega_j(k),T) + ikg_0^{(j)}(\omega_j(k),T)\right),$$

for $1 \leq j \leq n+1$.

3. "Solution" formula

Inverting the Fourier transform in the global relations we have

$$
\begin{aligned}
u^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k \\
&+ \frac{\sigma_j^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_j)-\omega_j(k)t}\left(h_1^{(j)}(\omega_j(k),t) + ikh_0^{(j)}(\omega_j(k),t)\right)\,\mathrm{d}k \\
&- \frac{\sigma_j^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_{j-1})-\omega_j(k)t}\left(g_1^{(j)}(\omega_j(k),t) + ikg_0^{(j)}(\omega_j(k),t)\right)\,\mathrm{d}k
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where $1 \le j \le n+1$.

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$$\begin{aligned} u^{(j)}(x,t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k \\ & + \frac{\sigma_j^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_j)-\omega_j(k)t}\left(h_1^{(j)}(\omega_j(k),t) + ik h_0^{(j)}(\omega_j(k),t)\right)\mathrm{d}k \\ & - \frac{\sigma_j^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_{j-1})-\omega_j(k)t}\left(g_1^{(j)}(\omega_j(k),t) + ik g_0^{(j)}(\omega_j(k),t)\right)\mathrm{d}k \end{aligned}$$

where $1 \le j \le n+1$.

4. Transformation to a common argument

We transform the global relations so that $g_\ell^{(j)}(\cdot, T)$ and $h_\ell^{(j)}(\cdot, T)$ depend on a common argument ν^2 through the change of variables $k = \nu/\sigma_j$:

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$$
\begin{aligned}
e^{\nu^2 T}\hat{u}^{(j)}\left(\frac{\nu}{\sigma_j}, T\right) = & \hat{u}_0^{(j)}\left(\frac{\nu}{\sigma_j}\right) + e^{-\frac{i\nu x_j}{\sigma_j}}\left(\sigma_j^2 h_1^{(j)} + i\sigma_j\nu h_0^{(j)}\right) \\
& - e^{-\frac{i\nu x_{j-1}}{\sigma_j}}\left(\sigma_j^2 g_1^{(j)} + i\sigma_j\nu g_0^{(j)}\right),
\end{aligned}
$$

where $1 \le j \le n+1$ and the arguments of $h_\ell^{(j)}$ and $g_\ell^{(j)}$ are (ν^2, T) .

4. Transformation to a common argument

Similarly, we transform the second and third integrals in the "solution",

$$\begin{aligned} u^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(\sigma_j h_1^{(j)}(\nu^2,T) + i\nu h_0^{(j)}(\nu^2,T)\right)\mathrm{d}\nu \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{i\nu(x-x_{j-1})}{\sigma_j}-\nu^2 t}\left(\sigma_j g_1^{(j)}(\nu^2,T) + i\nu g_0^{(j)}(\nu^2,T)\right)\mathrm{d}\nu, \end{aligned}$$

with $1 \le j \le n+1$ and $x_{j-1} < x < x_j$.

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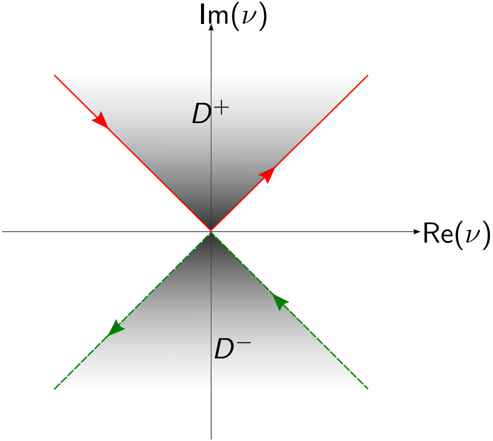

Figure: The regions $D^{\pm}$ for the heat equation.

4. Transformation to a common argument

$$
\begin{aligned}
u^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k \\
&-\frac{1}{2\pi}\int_{\partial D^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(\sigma_j h_1^{(j)}(\nu^2,T)+i\nu h_0^{(j)}(\nu^2,T)\right)\,\mathrm{d}\nu \\
&-\frac{1}{2\pi}\int_{\partial D^+} e^{\frac{i\nu(x-x_{j-1})}{\sigma_j}-\nu^2 t}\left(\sigma_j g_1^{(j)}(\nu^2,T)+i\nu g_0^{(j)}(\nu^2,T)\right)\,\mathrm{d}\nu,
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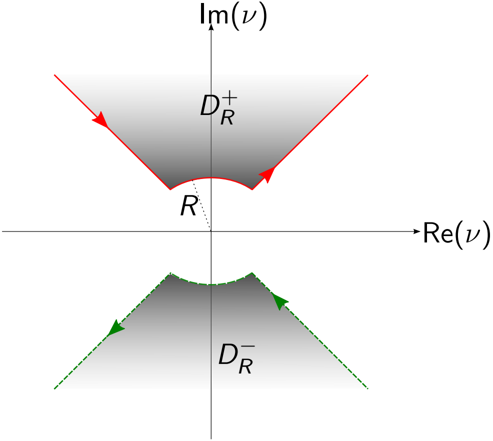

Figure: The regions $D_R^{\pm}$ for the heat equation.

4. Transformation to a common argument

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\begin{aligned}
u^{(j)}(x,t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\,\hat{u}_0^{(j)}(k)\,\mathrm{d}k \\
& - \frac{1}{2\pi}\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(\sigma_j h_1^{(j)}(\nu^2,T) + i\nu h_0^{(j)}(\nu^2,T)\right)\,\mathrm{d}\nu \\
& - \frac{1}{2\pi}\int_{\partial D_R^+} e^{\frac{i\nu(x-x_{j-1})}{\sigma_j}-\nu^2 t}\left(\sigma_j g_1^{(j)}(\nu^2,T) + i\nu g_0^{(j)}(\nu^2,T)\right)\,\mathrm{d}\nu,
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$$u^{(j)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k$$
$$-\frac{1}{2\pi}\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(\sigma_j h_1^{(j)}(\nu^2,T)+i\nu h_0^{(j)}(\nu^2,T)\right)\mathrm{d}\nu$$
$$-\frac{1}{2\pi}\int_{\partial D_R^+} e^{\frac{i\nu(x-x_{j-1})}{\sigma_j}-\nu^2 t}\left(\sigma_j g_1^{(j)}(\nu^2,T)+i\nu g_0^{(j)}(\nu^2,T)\right)\mathrm{d}\nu,$$

with $1 \leq j \leq n+1$ and $x_{j-1} < x < x_j$.
This solution is ineffective because it depends on the value of the function and its derivative evaluated at *all* the interfaces and boundaries.

4. Transformation to a common argument

Multiplying the boundary and interface conditions by $e^{\nu^2 t}$ and integrating the result from 0 to T with respect to t gives

$$\beta_1 g_0^{(1)}(\nu^2, T) + \beta_2 g_1^{(1)}(\nu^2, T) = \tilde{f}_1(\nu^2, T),$$
$$\beta_3 h_0^{(n+1)}(\nu^2, T) + \beta_4 h_1^{(n+1)}(\nu^2, T) = \tilde{f}_2(\nu^2, T),$$
$$\sigma_j^2 h_1^{(j)}(\nu^2, T) = H_j \left(g_0^{(j+1)}(\nu^2, T) - h_0^{(j)}(\nu^2, T) \right),$$
$$\sigma_{j+1}^2 g_1^{(j+1)}(\nu^2, T) = H_j \left(g_0^{(j+1)}(\nu^2, T) - h_0^{(j)}(\nu^2, T) \right),$$

for $1 \leq j \leq n$ where

$$\tilde{f}_1(\omega, t) = \int_0^t e^{\omega s} f_1(s) \, \mathrm{d}s,$$
$$\tilde{f}_2(\omega, t) = \int_0^t e^{\omega s} f_2(s) \, \mathrm{d}s.$$

4. Transformation to a common argument

$$u^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D_R^+} e^{\frac{i\nu x}{\sigma_1}-\nu^2 t}\left(\sigma_1 g_1^{(1)} + i\nu g_0^{(1)}\right)\,\mathrm{d}\nu$$

$$- \frac{1}{2\pi\sigma_1}\int_{\partial D_R^-} e^{\frac{i\nu(x-x_1)}{\sigma_1}-\nu^2 t}\left(H_1 g_0^{(2)} + (i\sigma_1\nu - H_1)\, h_0^{(1)}\right)\,\mathrm{d}\nu,$$

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$$u^{(j)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j(k)t}\hat{u}_0^{(j)}(k)\,\mathrm{d}k$$
$$- \frac{1}{2\pi\sigma_j}\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(H_j g_0^{(j+1)} + (i\sigma_j\nu - H_j)\,h_0^{(j)}\right)\mathrm{d}\nu$$
$$- \frac{1}{2\pi\sigma_j}\int_{\partial D_R^+} e^{\frac{i\nu(x-x_{j-1})}{\sigma_j}-\nu^2 t}\left((H_{j-1} + i\sigma_j\nu)\,g_0^{(j)} - H_{j-1}h_0^{(j-1)}\right)\mathrm{d}\nu,$$

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$$u^{(n+1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_{n+1}t}\hat{u}_0^{(n+1)}(k)\,\mathrm{d}k$$
$$- \frac{1}{2\pi}\int_{\partial D_R^-} e^{\frac{i\nu(x-x_{n+1})}{\sigma_{n+1}}-\nu^2 t}\left(\sigma_{n+1}h_1^{(n+1)} + i\nu h_0^{(n+1)}\right)\mathrm{d}\nu$$
$$- \frac{1}{2\pi\sigma_{n+1}}\int_{\partial D_R^+} e^{\frac{i\nu(x-x_n)}{\sigma_{n+1}}-\nu^2 t}\left((H_n + i\sigma_{n+1}\nu)\,g_0^{(n+1)} - H_n h_0^{(n)}\right)\mathrm{d}\nu,$$

5. Extra Global Relations

$$e^{\nu^2 T}\hat{u}^{(1)}\left(\frac{\nu}{\sigma_1}, T\right) = \hat{u}_0^{(1)}\left(\frac{\nu}{\sigma_1}\right) + e^{-\frac{i\nu x_1}{\sigma_1}}\left(H_1 g_0^{(2)} + (i\sigma_1\nu - H_1)h_0^{(1)}\right) - \sigma_1^2 g_1^{(1)} - i\sigma_1\nu g_0^{(1)},$$

$$e^{\nu^2 T}\hat{u}^{(j)}\left(\frac{\nu}{\sigma_j}, T\right) = \hat{u}_0^{(j)}\left(\frac{\nu}{\sigma_j}\right) + e^{-\frac{i\nu x_j}{\sigma_j}}\left(H_j g_0^{(j+1)} + (i\sigma_j\nu - H_j)h_0^{(j)}\right) + e^{-\frac{i\nu x_{j-1}}{\sigma_j}}\left(H_{j-1} h_0^{(j-1)} - (H_{j-1} + i\sigma_j\nu)g_0^{(j)}\right),$$

$$e^{\nu^2 T}\hat{u}^{(n+1)}\left(\frac{\nu}{\sigma_{n+1}}, T\right) = \hat{u}_0^{(n+1)}\left(\frac{\nu}{\sigma_{n+1}}\right) + e^{-\frac{i\nu x_{n+1}}{\sigma_{n+1}}}\left(\sigma_{n+1}^2 h_1^{(n+1)} + i\sigma_{n+1}\nu h_0^{(n+1)}\right) + e^{-\frac{i\nu x_n}{\sigma_{n+1}}}\left(H_n h_0^{(n)} - (H_n + i\sigma_{n+1}\nu)g_0^{(n+1)}\right),$$

5. Extra Global Relations

$$
\begin{aligned}
e^{\nu^2 T}\hat{u}^{(1)}\left(\frac{-\nu}{\sigma_1}, T\right) =& \hat{u}_0^{(1)}\left(\frac{-\nu}{\sigma_1}\right) + e^{\frac{i\nu x_1}{\sigma_1}}\left(H_1 g_0^{(2)} - (i\sigma_1\nu + H_1)h_0^{(1)}\right) \\
& - \sigma_1^2 g_1^{(1)} + i\sigma_1\nu g_0^{(1)}, \\
e^{\nu^2 T}\hat{u}^{(j)}\left(\frac{-\nu}{\sigma_j}, T\right) =& \hat{u}_0^{(j)}\left(\frac{-\nu}{\sigma_j}\right) + e^{\frac{i\nu x_j}{\sigma_j}}\left(H_j g_0^{(j+1)} - (i\sigma_j\nu + H_j)h_0^{(j)}\right) \\
& + e^{\frac{i\nu x_{j-1}}{\sigma_j}}\left(H_{j-1} h_0^{(j-1)} - (H_{j-1} - i\sigma_j\nu)g_0^{(j)}\right), \\
e^{\nu^2 T}\hat{u}^{(n+1)}\left(\frac{-\nu}{\sigma_{n+1}}, T\right) =& \hat{u}_0^{(n+1)}\left(\frac{-\nu}{\sigma_{n+1}}\right) + e^{\frac{i\nu x_{n+1}}{\sigma_{n+1}}}\left(\sigma_{n+1}^2 h_1^{(n+1)} - i\sigma_{n+1}\nu h_0^{(n+1)}\right) \\
& + e^{\frac{i\nu x_n}{\sigma_{n+1}}}\left(H_n h_0^{(n)} - (H_n - i\sigma_{n+1}\nu)g_0^{(n+1)}\right),
\end{aligned}
$$

5. Extra Global Relations

- Our "solution" involves $2n+4$ unknown functions $g_0^{(j)}(\nu^2, T)$, $h_0^{(j)}(\nu^2, T)$, $g_1^{(1)}(\nu^2, T)$, $h_1^{(n+1)}(\nu^2, t)$ for $1 \leq j \leq n+1$.

5. Extra Global Relations

- Our "solution" involves $2n+4$ unknown functions $g_0^{(j)}(\nu^2, T)$, $h_0^{(j)}(\nu^2, T)$, $g_1^{(1)}(\nu^2, T)$, $h_1^{(n+1)}(\nu^2, t)$ for $1 \leq j \leq n+1$.
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- These same unknown functions are related through the $2n+2$ global relations and the transformed boundary conditions.
- Solving this linear system for the unknown functions amounts to solving a $(2n+4) \times (2n+4)$ matrix problem.

5. Extra Global Relations

$$\mathcal{A}X = \mathcal{Y} + Y$$

where

$$X = \left(g_1^{(1)}, g_0^{(1)}, \ldots, g_0^{(n+1)}, h_0^{(1)}, \ldots, h_0^{(n+1)}, h_1^{(n+1)}\right)^\top,$$

$$Y = -\left(-\tilde{f}_1, \hat{u}_0^{(1)}\left(\frac{\nu}{\sigma_1}\right), \cdots, \hat{u}_0^{(n+1)}\left(\frac{\nu}{\sigma_{n+1}}\right), \hat{u}_0^{(1)}\left(\frac{-\nu}{\sigma_1}\right), \cdots, \hat{u}_0^{(n+1)}\left(\frac{-\nu}{\sigma_{n+1}}\right), -\tilde{f}_2\right)^\top,$$

$$\mathcal{Y} = e^{\nu^2 T}\left(0, \hat{u}^{(1)}\left(\frac{\nu}{\sigma_1}, T\right), \cdots, \hat{u}^{(n+1)}\left(\frac{\nu}{\sigma_{n+1}}, T\right), \hat{u}^{(1)}\left(\frac{-\nu}{\sigma_1}, T\right), \cdots, \hat{u}^{(n+1)}\left(\frac{-\nu}{\sigma_{n+1}}, T\right), 0\right)^\top,$$

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5. Extra Global Relations

$$\mathcal{A} = \left(\begin{array}{c:c} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \hdashline \mathcal{A}_{21} & \mathcal{A}_{22} \end{array}\right).$$

5. Extra Global Relations

$$
\mathcal{A}_{11} = \begin{pmatrix}
\beta_2 & \beta_1 & & & \\
-\sigma_1^2 & -i\sigma_1\nu & H_1 e^{-i\frac{\nu x_1}{\sigma_1}} & & & \\
0 & 0 & -(H_1 + i\sigma_2\nu)e^{-i\frac{\nu x_1}{\sigma_2}} & H_2 e^{-i\frac{\nu x_2}{\sigma_2}} & & \\
\vdots & & \ddots & \ddots & & \\
\vdots & & & -(H_{n-1} + i\sigma_n\nu)e^{-i\frac{\nu x_{n-1}}{\sigma_n}} & H_n e^{-i\frac{\nu x_n}{\sigma_n}} \\
0 & \dots & \dots & 0 & -(H_n + i\sigma_{n+1}\nu)e^{-i\frac{\nu x_n}{\sigma_{n+1}}}
\end{pmatrix},
$$

$$
\mathcal{A}_{12} = \begin{pmatrix}
0 & \dots & \dots & 0 & 0 \\
(i\sigma_1\nu - H_1)e^{-i\frac{\nu x_1}{\sigma_1}} & & & & 0 \\
H_1 e^{-i\frac{\nu x_1}{\sigma_2}} & (i\sigma_2\nu - H_2)e^{-i\frac{\nu x_2}{\sigma_2}} & 0 & \dots & 0 \\
\ddots & \ddots & & & \vdots \\
& H_{n-1} e^{-i\frac{\nu x_{n-1}}{\sigma_n}} & (i\sigma_n\nu - H_n)e^{-i\frac{\nu x_n}{\sigma_n}} & & \vdots \\
& & H_n e^{-i\frac{\nu x_n}{\sigma_{n+1}}} & i\sigma_{n+1}\nu e^{-i\frac{\nu x_{n+1}}{\sigma_{n+1}}} & \sigma_{n+1}^2 e^{-i\frac{\nu x_{n+1}}{\sigma_{n+1}}}
\end{pmatrix},
$$

5. Extra Global Relations

$$\mathcal{A}_{21} = \begin{pmatrix} -\sigma_1^2 & i\sigma_1\nu & H_1 e^{i\frac{\nu x_1}{\sigma_1}} & 0 & & \\ 0 & 0 & (i\sigma_2\nu - H_1)e^{i\frac{\nu x_1}{\sigma_2}} & H_2 e^{i\frac{\nu x_2}{\sigma_2}} & & \\ \vdots & & \ddots & & \ddots & \\ \vdots & & & (i\sigma_n\nu - H_{n-1})e^{i\frac{\nu x_{n-1}}{\sigma_n}} & & H_n e^{i\frac{\nu x_n}{\sigma_n}} \\ 0 & & & & & (i\sigma_{n+1}\nu - H_n)e^{i\frac{\nu x_n}{\sigma_{n+1}}} \\ 0 & \dots & \dots & \dots & & 0 \end{pmatrix},$$

$$\mathcal{A}_{22} = \begin{pmatrix} -(i\sigma_1\nu + H_1)e^{i\frac{\nu x_1}{\sigma_1}} & & & & \\ H_1 e^{i\frac{\nu x_1}{\sigma_2}} & -(i\sigma_2\nu + H_2)e^{i\frac{\nu x_2}{\sigma_2}} & & & \\ \ddots & \ddots & & & \\ & H_{n-1}e^{i\frac{\nu x_{n-1}}{\sigma_n}} & -(i\sigma_n\nu + H_n)e^{i\frac{\nu x_n}{\sigma_n}} & & \\ & & H_n e^{i\frac{\nu x_n}{\sigma_{n+1}}} & -i\sigma_{n+1}\nu e^{i\frac{\nu x_{n+1}}{\sigma_{n+1}}} & \sigma_{n+1}^2 e^{i\frac{\nu x_{n+1}}{\sigma_{n+1}}} \\ & & & \beta_3 & \beta_4 \end{pmatrix},$$

5. Extra Global Relations

$$\mathcal{A}_{21} = \begin{pmatrix} -\sigma_1^2 & i\sigma_1\nu & H_1 e^{i\frac{\nu x_1}{\sigma_1}} & 0 & \\ 0 & 0 & (i\sigma_2\nu - H_1)e^{i\frac{\nu x_1}{\sigma_2}} & H_2 e^{i\frac{\nu x_2}{\sigma_2}} & \\ \vdots & & \ddots & & \ddots \\ \vdots & & & (i\sigma_n\nu - H_{n-1})e^{i\frac{\nu x_{n-1}}{\sigma_n}} & H_n e^{i\frac{\nu x_n}{\sigma_n}} \\ 0 & & & & (i\sigma_{n+1}\nu - H_n)e^{i\frac{\nu x_n}{\sigma_{n+1}}} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix},$$

$$\mathcal{A}_{22} = \begin{pmatrix} -(i\sigma_1\nu + H_1)e^{i\frac{\nu x_1}{\sigma_1}} & & & & \\ H_1 e^{i\frac{\nu x_1}{\sigma_2}} & -(i\sigma_2\nu + H_2)e^{i\frac{\nu x_2}{\sigma_2}} & & & \\ \ddots & \ddots & & & \\ & H_{n-1}e^{i\frac{\nu x_{n-1}}{\sigma_n}} & -(i\sigma_n\nu + H_n)e^{i\frac{\nu x_n}{\sigma_n}} & & \\ & & H_n e^{i\frac{\nu x_n}{\sigma_{n+1}}} & -i\sigma_{n+1}\nu e^{i\frac{\nu x_{n+1}}{\sigma_{n+1}}} & \sigma_{n+1}^2 e^{i\frac{\nu x_{n+1}}{\sigma_{n+1}}} \\ & & & \beta_3 & \beta_4 \end{pmatrix},$$

$$\mathcal{A}X = Y + \mathcal{Y}$$

5. Extra Global Relations

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\mathcal{A}_{21} = \begin{pmatrix}
-\sigma_1^2 & i\sigma_1\nu & H_1 e^{i\frac{\nu x_1}{\sigma_1}} & 0 & \\
0 & 0 & (i\sigma_2\nu - H_1)e^{i\frac{\nu x_1}{\sigma_2}} & H_2 e^{i\frac{\nu x_2}{\sigma_2}} & \\
\vdots & & \ddots & \ddots & \\
\vdots & & & (i\sigma_n\nu - H_{n-1})e^{i\frac{\nu x_{n-1}}{\sigma_n}} & H_n e^{i\frac{\nu x_n}{\sigma_n}} \\
0 & & & & (i\sigma_{n+1}\nu - H_n)e^{i\frac{\nu x_n}{\sigma_{n+1}}} \\
0 & \dots & \dots & \dots & 0
\end{pmatrix},
$$

$$
\mathcal{A}_{22} = \begin{pmatrix}
-(i\sigma_1\nu + H_1)e^{i\frac{\nu x_1}{\sigma_1}} & & & & \\
H_1 e^{i\frac{\nu x_1}{\sigma_2}} & -(i\sigma_2\nu + H_2)e^{i\frac{\nu x_2}{\sigma_2}} & & & \\
\ddots & \ddots & & & \\
& H_{n-1}e^{i\frac{\nu x_{n-1}}{\sigma_n}} & -(i\sigma_n\nu + H_n)e^{i\frac{\nu x_n}{\sigma_n}} & & \\
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$$
\mathcal{A}X = Y + \mathcal{Y}
$$

Let $\mathcal{A}_j$ be the matrix $\mathcal{A}$ with the j^{th} column replaced by $\mathcal{Y}$. Factor $\mathcal{A}_j = \mathcal{A}^{(L,\pm)}\mathcal{A}_j^{(M,\pm)}\mathcal{A}_j^R$ where $\mathcal{A}_j^R$ is the $(2n+4)\times(2n+4)$ identity matrix with the (j,j) entry replaced by $e^{\nu^2 T}$.

$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}-\nu^2 t}\left(H_j g_0^{(j+1)} + (i\sigma_j\nu - H_j)\, h_0^{(j)}\right)\mathrm{d}\nu,$$

and

$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_n)}{\sigma_n}-\nu^2 t}\left(\sigma_n h_1^{(n)} + i\nu h_0^{(n)}\right)\mathrm{d}\nu,$$

6. Solution

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$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}+\nu^2(T-t)} \left(H_j \frac{\det(\mathcal{A}_{j+2}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} + (i\sigma_j\nu - H_j)\frac{\det(\mathcal{A}_{n+j+1}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} \right) \mathrm{d}\nu.$$

and

$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_n)}{\sigma_n}+\nu^2(T-t)} \left(\sigma_n \frac{\det(\mathcal{A}_{2n+2}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} + i\nu\frac{\det(\mathcal{A}_{2n+1}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} \right) \mathrm{d}\nu.$$

6. Solution

Expanding the determinant of $\mathcal{A}_j^{(M,-)}$ along the j^{th} column we have

$$\frac{\det(\mathcal{A}_{j+2}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} = \sum_{\ell=1}^{n} \left(\alpha_\ell(\nu) e^{\frac{-i\nu x_\ell}{\sigma_\ell}} \hat{u}^{(\ell)} \left(\frac{\nu}{\sigma_\ell}, T \right) + \beta_\ell(\nu) e^{\frac{i\nu x_{\ell-1}}{\sigma_\ell}} \hat{u}^{(\ell)} \left(\frac{-\nu}{\sigma_\ell}, T \right) \right)$$

where $\alpha_\ell(\nu)$ and $\beta_\ell(\nu)$ are $\mathcal{O}(\nu^0)$ for large ν and $x_{j-1} < x < x_j$.

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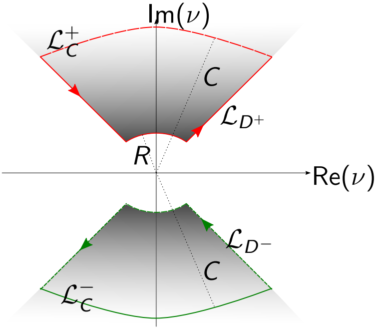
where $\alpha_\ell(\nu)$ and $\beta_\ell(\nu)$ are $\mathcal{O}(\nu^0)$ for large ν and $x_{j-1} < x < x_j$.

When multiplied by the exponential in the "solution," these terms decay exponentially fast for $|\nu| \to \infty$ from within D_R^-!

6. Solution

$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_j)}{\sigma_j}+\nu^2(T-t)} \left(H_j \frac{\det(\mathcal{A}_{j+2}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} + (i\sigma_j\nu - H_j) \frac{\det(\mathcal{A}_{n+j+1}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} \right) \mathrm{d}\nu,$$

$$\int_{\partial D_R^-} e^{\frac{i\nu(x-x_n)}{\sigma_n}+\nu^2(T-t)} \left(\sigma_n \frac{\det(\mathcal{A}_{2n+2}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} + i\nu \frac{\det(\mathcal{A}_{2n+1}^{(M,-)})}{\det(\mathcal{A}^{(M,-)})} \right) \mathrm{d}\nu.$$



6. Solution

$g_1^{(1)}(\nu^2, T), g_0^{(j)}(\nu^2, T), h_0^{(j)}(\nu^2, T)$, and $h_1^{(n)}(\nu^2, T)$ for $1 \leq j \leq n+1$ are found by solving

$$\mathcal{A}X = Y,$$

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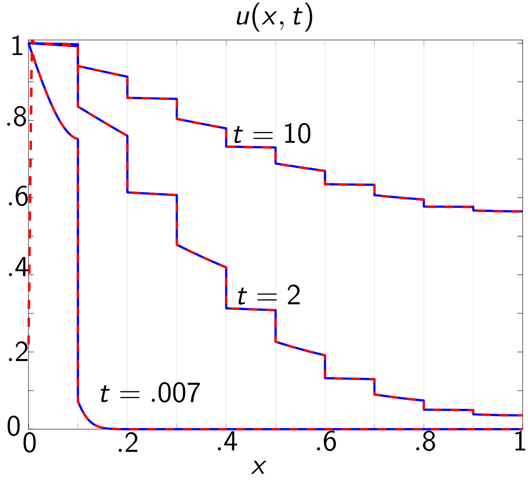
`https://github.com/nsheils/UTM_Heat`

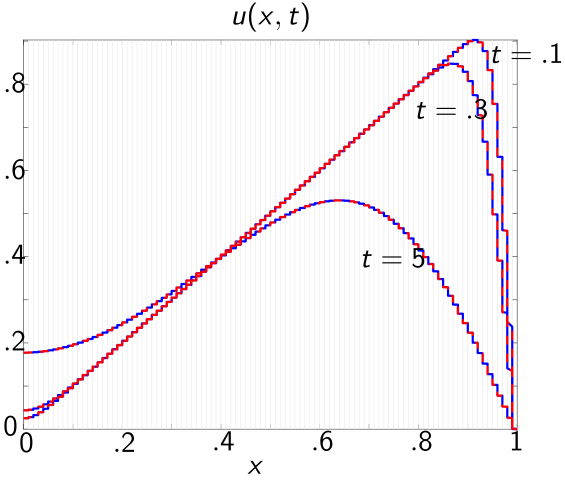
Example 1

The solution to the heat equation with 9 interfaces and alternating diffusivities 1 and $\sqrt{.1}$, $u(x,0) = 0$, $u(0,t) = 1$, and $u_x(1,t) = 0$ and we assume imperfect thermal contact with $H_j = 1/2$ for $j = 1, \cdots, 9$.



Example 2

In this example $n = 199$, $\sigma_j = \sqrt{1.1 + \sin(j)}$ for $j = 1, \cdots, n$. We let $u(x, 0) = x$, $u_x(0, t) = 0$ and $u(1, t) = 0$ The x_j are evenly spaced and we assume imperfect thermal contact with $H_j = 1/2$ for $j = 1, \cdots, n$.



Summary

1. Local relation, dispersion relations $\omega_j(k)$
2. Global relation
3. "Solution reconstruction," with deformed contour
4. Transformation to a common argument
5. Extra global relations, using symmetries
6. Solution, after eliminating unknown boundary and interface conditions

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1. Local relation, dispersion relations $\omega_j(k)$
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These steps are the same for any initial value problem with piecewise-constant coefficients

Conclusions

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- This is a general method.
- The use of the discrete symmetries of the dispersion relation is an important aspect of the UTM. All nontrivial discrete symmetries are needed to derive the final expression for the interface problem.
- The UTM can be used to solve interface problems that are not solvable using standard methods.

References

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Applied Mathematical Modelling, June 2017.

https://github.com/nsheils/UTM_Heat

More on the UTM:
http://unifiedmethod.azurewebsites.net

Questions

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