Sharp bounds for oscillatory integral operators via polynomial partitioning, II

Joint work with L. Guth and J. Hickman

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UC Berkeley

Harmonic Analysis and its interactions, in honour of Tony Carbery

Edinburgh, 18 July 2017

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If f is concentrated on wave packets tangential to a $(k-1)$-dim variety, then

$\|T^\lambda f\|_{BL^p_{k,A}(B_R)}$ is negligible.

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3. Lowest $\bar{p}(k,n)$ that fits in above interval: given for $k = \frac{n}{2} + 1$ when n is even, $k = \frac{n+1}{2}$ when n is odd.

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Wall domination: Mass concentrated $R^{1/2+\delta_m}$-close to Z^l, for some $l < m$.

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$$\lesssim \|f\|_2^2.$$

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Then, g **cannot** concentrate on wave packets in a much thinner strip inside $N_{R^{1/2+\epsilon^m}}(Z^m)$.

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Let g be $R^{-1/2+\epsilon^m}$-tangential to Z^m.

Then, g **cannot** concentrate on wave packets in a much thinner strip inside $N_{R^{1/2+\epsilon^m}}(Z^m)$.

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Thank you!