Maximal functions associated to 2-hypersurfaces with height < 2

Detlef Müller

University of Kiel
Mathematisches Seminar

Joint work with S. Buschenhenke, S. Dendrinos and I. A. Ikromov

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I. Introduction: Maximal functions associated to hypersurfaces

$S =$ smooth, finite type hypersurface in $\mathbb{R}^n$,

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$$\begin{aligned} A_t f(x) &:= \int_S f(x - ty)\, d\mu(y), \qquad t > 0 \\ \mathcal{M}f(x) = \mathcal{M}_S f(x) &:= \sup_{t>0} |A_t f(x)| \end{aligned}$$

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QUESTION:

Find the maximal range of p's for which

(1) $$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Spherical maximal function and dimension free estimates

- [Stein '76] If $S = S^{n-1}$, $n \geq 3$, is the unit sphere, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ iff $p > n/(n-1)$.
- [Bourgain'86] Same for the circular maximal function ($n = 2$).
- Using this + method of rotation, Stein deduced dimension free estimates for the centered Hardy-Littlewood maximal operator on $\mathbb{R}^n$.
- [Bourgain'86] Dimension free estimates for Hardy-Littlewood maximal operators associated to centrally symmetric convex bodies when $p > 2$
- [Bourgain; Carbery '86] Same for $p > 3/2$.
- [M.'90] A geometric bound for such maximal operators, which leads to dimension free estimates for l^q-balls when $1 \leq q < \infty$.
- [Bourgain' 2012] Dimension free estimates for the cubic maximal operator (l^∞-balls)

Back to $\mathcal{M}_S$

Stein's article stimulated a lot of investigations, in particular for

(a) Convex hypersurfaces S of finite line type [Bruna/Nagel/Wainger, Nagel/Seeger/Wainger, Iosevich, Sawyer, Seeger, ...]

Based on "method of caps" introduced by Bruna/Nagel/Wainger - not available anymore for non-convex surfaces.

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Note:

The problem can be localized to sufficiently small neighborhoods $U \cap S$ of given points $x^0 \in S$

II. 2-hypersurfaces $S \subset \mathbb{R}^3$

PROGRAMME:

For **any** finite type 2-hypersurface S, describe the maximal range of p's for which $\mathcal{M}_S$ is L^p-bounded, if possible, in terms of Newton diagrams!

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$\Longrightarrow$ Given $x^0 \in S$, after applying a **linear** change of coordinates in $\mathbb{R}^3$ we may assume $x^0 = (0, 0, 1)$, and

$$S = \{(x_1, x_2, 1 + \phi(x_1, x_2) : (x_1, x_2) \in \Omega\}$$

$$\phi(0,0) = 0, \nabla\phi(0,0) = 0.$$

Newton polyhedra

$$\phi(x_1, x_2) \sim \sum_{j,k=0}^{\infty} c_{jk} x_1^j x_2^k$$

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- Newton distance : $d = d(\phi)$ is given by the coordinate d of the point (d,d) at which the bisectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron.

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$$\phi_{\mathrm{pr}}(x_1, x_2) := \sum_{(j,k) \in \pi(\phi)} c_{jk} x_1^j x_2^k$$

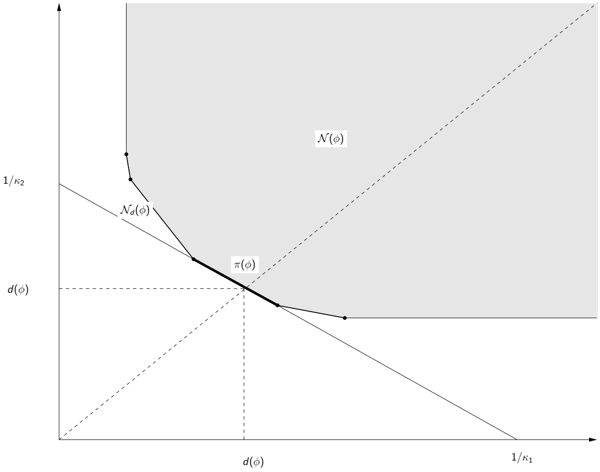

Figure: Newton polyhedron

Adapted coordinates

- Height of ϕ :

$$h = h(\phi) := \sup\{d_x\},$$

where the supremum is taken over all local smooth coordinate systems x at the origin, and where d_x is the Newton distance of ϕ when expressed in the coordinates x.

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Example. Let
$$\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell.$$
If $\ell > mn$, the coordinates are not adapted. Adapted coordinates are then $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by
$$\phi^a(y) = y_2^n + y_1^\ell.$$

- If $\pi(\phi)$ is a compact edge, the principal line is the unique line L such that
$$L = \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\} \supset \pi(\phi).$$
- principal weight $\kappa := (\kappa_1, \kappa_2)$; $\qquad |\kappa| := \kappa_1 + \kappa_2$
- associated dilations $\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2),\ r > 0$
- homogeneity: $\qquad \phi_{\mathrm{pr}}(\delta_r x) = r \phi_{\mathrm{pr}}(x)$

Previous results: $h \geq 2$

Theorem (Ikromov, Kempe, M. '2010)

Assume that $h \geq 2$, and that the tranversality condition holds true. If the measure $d\mu = \rho d\sigma$ is supported in a sufficiently small neighborhood of x^0, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^3)$ whenever $p > h$. Moreover, if $\rho(x^0) \neq 0$, then this conditions is also necessary, if S is analytic, with the possible exception of $p = h$ when S is non-analytic.

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Theorem (Zimmermann '2014)

Assume that $h \geq 2$ *and* $x^0 = 0 \in S$, *where* S *is analytic. If the measure* $d\mu = \rho d\sigma$ *is supported in a sufficiently small neighborhood of* 0, *then* $\mathcal{M}$ *is bounded on* $L^p(\mathbb{R}^3)$ *whenever* $p > 2$.

Partial results also by [Greenblatt], by using a different approach.

III. On the case $h < 2$:

Normal forms when $h < 2$ and 0 is a degenerate critical point of ϕ

By means of a linear change of coordinates, ϕ can then be assumed to be in one of the following forms near 0 (corresponding to Arnold's classification of singularities):

Case 1 (Type A). $\mathrm{rank}\, D^2\phi(0,0) = 1.$

(2) $$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + x_1^n \beta(x_1),$$

where $\beta(0) \neq 0$, and

- $b(0,0) \neq 0$
- the "principal root jet" ψ is either flat at 0, or

$$\psi(x_1) = x_1^m \omega(x_1), \quad \omega(0) \neq 0, m \geq 2,$$

- $n \geq 3$ (type A_{n-1})

Case 2. $\operatorname{rank} D^2\phi(0,0) = 0$. We distinguish two subcases:

(i) Type D. ϕ is still of the form (2), but now

- $b(0,0) = 0$
- $b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2), \qquad b_1(0,0) \neq 0$
- $n \geq 3$ (type D_{n+1})

(ii) Type E. $\phi = \phi_{\mathrm{pr}} + \phi_r$, where the principal part ϕ_{pr} of ϕ is one of the following forms:

- $\phi_{\mathrm{pr}}(x_1, x_2) = x_2^3 \pm x_1^4,$ (type E_6)
- $\phi_{\mathrm{pr}}(x_1, x_2) = x_2^3 + x_1^3 x_2,$ (type E_7)
- $\phi_{\mathrm{pr}}(x_1, x_2) = x_2^3 + x_1^5,$ (type E_8)

Main Theorem

Theorem (1)

Assume that S *is a smooth, finite-type hypersurface in* $\mathbb{R}^3$ *satisfying the transversality assumption, and let* $x^0 \in S$ *be a given point at which* $h < 2$ *and both principal curvatures of* S *do vanish (Case 2). If the measure* $d\mu = \rho d\sigma$ *is supported in a sufficiently small neighborhood of* x^0*, then* $\mathcal{M}$ *is bounded on* $L^p(\mathbb{R}^3)$ *whenever* $p > h$*. This condition is also necessary if* $\rho(x^0) \neq 0$.

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Most difficult case: D_4- type singularities !

III.2 Sketch of proof for D_4- type singularities

Here, without loss of generality,

(3) $$\phi_{\mathrm{pr}}(x_1, x_2) = x_1 x_2^2 \pm x_1^3 \qquad \text{(type } D_4^{\pm}\text{)}$$

- The coordinates (x_1, x_2) are already adapted
- $\kappa_1 = \kappa_2 = 1/3; \qquad h = 3/2$
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1. Step (standard): Dyadic decomposition of $\Omega \subset \mathbb{R}^2$ by means of the dilations δ_r :

(4) $$\|\mathcal{M}\|_{L^p \to L^p} \leq \sum_{k=k_0}^{\infty} 2^{-|\kappa|k} \|\mathcal{M}_k\|_{L^p \to L^p},$$

where (essentially)

$$\mathcal{M}_k f(y, y_3) := \sup_{t>0} |f * (\mu_k)_t(y, y_3)|,$$

$$\int f \, d\mu_k := \int_{|x|\sim 1} f(x, 2^k + \phi^k(x)) \, \chi_1(x) dx,$$

with

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Lemma (A)

Assume that

$$(5) \qquad \left| \int e^{i(\xi\cdot x + \xi_3 \phi^k(x))}\, \chi_1(x) dx \right| \le C \frac{C}{(1+|\xi_3|)^\gamma}, \qquad (\xi, \xi_3) \in \mathbb{R}^3,$$

and that $1 \ge \gamma > 1/2$. *Then for every* $p > 1 + 1/(2\gamma)$

$$\|\mathcal{M}_k\|_{p\to p} \le C_p T^{\frac{1}{p}}.$$

Reduction to A_2-type singularities

By (4), $\mathrm{Hess}(\phi_{\mathrm{pr}})(x_1, x_2) = \mp 12x_1^2 - 4x_2^2$.

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(a) If ϕ is of type D_4^-, $\mathrm{Hess}(\phi_{\mathrm{pr}})(x) \neq 0$ when $|x| \sim 1$, so by stationary phase in x_1 and x_2 we get (5), with $\gamma = 1$. Thus, for $p > 1 + 1/2 = 3/2 = h$ we get $\|\mathcal{M}_k\|_{p \to p} \leq C_p 2^{k/p}$, and so by (4)

$$\|\mathcal{M}\|_{L^p \to L^p} \leq \sum_{k=k_0}^{\infty} 2^{-\frac{2}{3}k} 2^{k/p} < \infty.$$

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(b) Much harder is the case where ϕ is of type D_4^+ : Here, if $|x^0| \sim 1$ and $\sqrt{3}|x_2^0| = |x_1^0|$, then $\mathrm{Hess}(\phi_{\mathrm{pr}})(x^0) = 0$. By means of an affine change of coordinates, we may then reduce to a phase of the form

$$\tilde{\phi}_{\mathrm{pr}}(y_1, y_2) = y_2^2 + y_1^3, \quad |(y_1, y_2)| \ll 1 \quad (A_2\text{-type singularity! })$$

CONCLUSION: We need estimates of $\mathcal{M}_S$ for families of surfaces S of A_2-type, which are stable under small perturbations of S!

IV: A_2-type $\phi = b(x_1, x_2)(x_2 - x_1^m \omega(x_1))^2 + x_1^3 \beta(x_1)$

Here, we are in Case 1 of our normal forms:

- $\mathrm{rank}\, D^2\phi(0,0) = 1$
- $h = d = 1/(1/2 + 1/3) = 6/5 > 3/2$

The following main result is even stable under small perturbations of ϕ:

Theorem (2)

For $T \gg 1$ let

$$\mathcal{M}_T f(y, y_3) := \sup_{t>0} \Big| \int_{\mathbb{R}^2} f(y - tx, y_3 - t(T + \phi(x))\eta(x)\, dx \Big| =: \sup_{t>0} |f * (\mu_T)_t(y, y_3)|.$$

Then for $p > 3/2$,

$$\|\mathcal{M}_T\|_{p\to p} \le C_{\delta,p} T^{\frac{1}{p}+\delta},$$

for every given $\delta > 0$.

IV.1 Airy analysis

1. Step: Dyadic frequency decompositions ($\lambda = 2^j \gg 1$) of μ_T into μ_T^λ's (modulo small error) given by

$$\widehat{\mu_T^\lambda}(\xi) := \chi_0\left(\frac{\xi_1}{\lambda}, \frac{\xi_2}{\lambda}\right)\chi_1\left(\frac{\xi_3}{\lambda}\right)\widehat{\mu_T}(\xi).$$

where $|(\xi_1, \xi_2)| \ll 1$ on $\operatorname{supp}\chi_0$ and $|\xi_3| \sim 1$ on $\operatorname{supp}\chi_1$. For λ fixed, write

$$\xi_3 = \lambda s_3, \quad \xi_1 = \lambda s_3 s_1, \quad \xi_2 = \lambda s_3 s_2, \tag{4}$$

and put $s' := (s_1, s_2), s := (s', s_3)$. Then we have

$$|s_3| \sim 1 \quad \text{and} \quad |s'| \ll 1$$

on the support of $\widehat{\mu_T^\lambda}$.

Then,

$$\widehat{\mu_T^\lambda}(\xi) = e^{-iT\xi_3}\chi_0(s_3 s')\chi_1(s_3)J(\lambda, s), \tag{5}$$

where $J(\lambda, s) = J(\lambda)$ denotes the oscillatory integral

$$J(\lambda, s) := \int_{\mathbb{R}^2} e^{-i\lambda s_3 \Phi(x, s', \sigma)} a_0(x)\, dx,$$

with complete phase

$$\Phi(x, s') := s_1 x_1 + s_2 x_2 + \phi(x).$$

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Changing coordinates $x_2 \mapsto x_2 + x_1^m \omega(x_1)$ and applying then stationary phase in x_2 leads to 1-dim. oscillatory integral, with phase roughly of the form

$$\tilde{\Psi}(x, s') = s_2^2 b_1(x_1, s_2) + s_1 x_1 + s_2 x_1^m \omega(x_1) + x_1^3 \beta(x_1).$$

Find critical point x_1^c of $\partial_{x_1}\tilde{\Psi}$ and translate coordinate x_1 by x_1^c to arrive at new phase of the form (where $B_3 \neq 0$)

$$\Psi(x_1, s') = B_0(s') + B_1(s')x_1 + x_1^3 B_3(x_1,\, s_2).$$

2. Step: Perform dyadic frequency decomposition w.r. to the Airy cone

$$\Gamma := \{B_1(s') = 0\} = \{\xi : B_1\big(\frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\big) = 0\}.$$

- This leads to complicated oscillatory integrals which need to be estimated
- At the end of the day, for every "dyadic Airy piece" arising from this decomposition, the main contribution to the maximal operator comes from a small, curved box at distance $T \gg 1$. The corresponding maximal operator can be estimated near $p = 1$ by means of the following

Lemma (B) (Variation on Hardy-Littlewood's maximal operator)

Let A be an open subset of $\mathbb{R}^n$ contained in the annulus $T \leq |x| < 2T$, where $T > 0$, and let

$$\mathcal{M}_A f(x) := \sup_{t>0} \int_A |f(x+ty)|\, dy, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Then, for $1 < p \leq \infty$, we have that

$$\|\mathcal{M}_A f\|_{L^p \to L^p} \leq C_p T^n |\pi(A)|, \tag{6}$$

where $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ the spherical projection onto the unit sphere S^{n-1}, given by $\pi(x) := x/|x|$.

Remarks:

- Proof by Whitney decomposition of $\pi(A)$
- Is only optimal as $p \to 1$ (more precisely, for weak (1,1)-estimate).

What remains open: A_{n-1}-type singularities, $n \geq 3$)

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + x_1^n \beta(x_1),$$

Examples

(a) [Iosevich/Sawyer/Seeger] (Convex case; notion of multitype)

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$$p > p_n := \max\left\{\frac{3}{2}, \frac{2n}{n+1}, h\right\} \tag{8}$$

These number are related to the order of contact of S with points, lines, respectively hyperplanes! Note that here $h < 2n/(n+1)$ - so h does not matter! $n = 3$ formally matches with our result on A_2! The case $n \geq 3$ in (7) follows easily from Lemma (A).

Perturbation terms change the picture!

(b) [Iosevich/Sawyer/Seeger]

(9) $\phi = (x_2^2 + x_1^4) + x_1^2 x_2^2$ (A_5-singularity; convex surface)

Here, condition (8) would require $p > 8/5 (> 3/2)$, but indeed $p > 3/2$ is sufficient!

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Here the condition

(11) $$p > p_n^c := \max\left\{\frac{3}{2}, \frac{2(n+1)}{n+3}\right\}$$

is necessary, and I believe also sufficient! Note that $p_n^c < p_n$.
(Relations to reverse square function conjecture for cone!)

Thanks for your attentions, and