On the construction of entire functions in the Speiser class

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Edinburgh, 23 May 2013

1. Motivation

Question: Given a domain G in the plane, does there exist an entire function in class $\mathcal{S}$ which is bounded on $\mathbb{C} \setminus \overline{G}$?

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One possibility to construct functions in class $\mathcal{S}$ with a given property is *quasiconformal folding*, a method introduced by C. Bishop in 2011.

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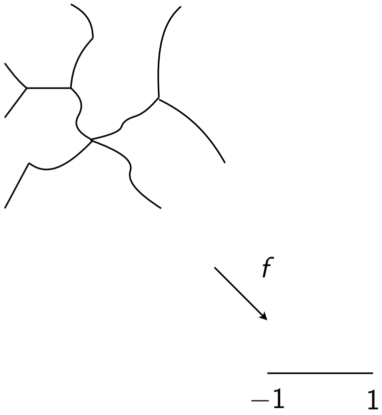

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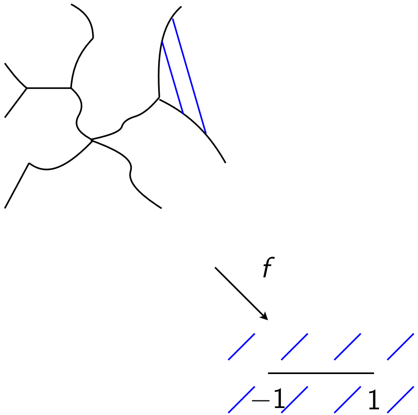

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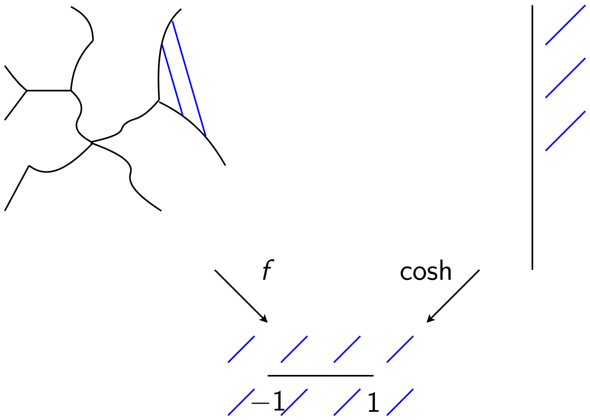

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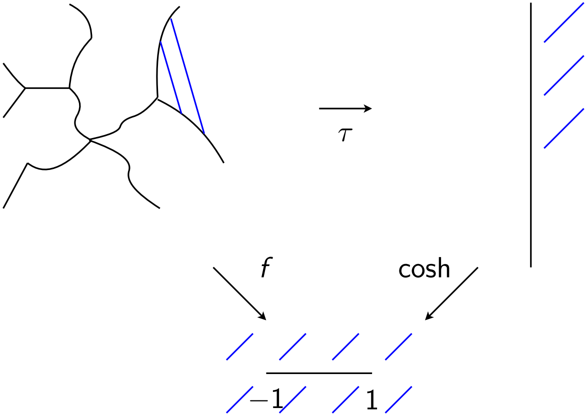

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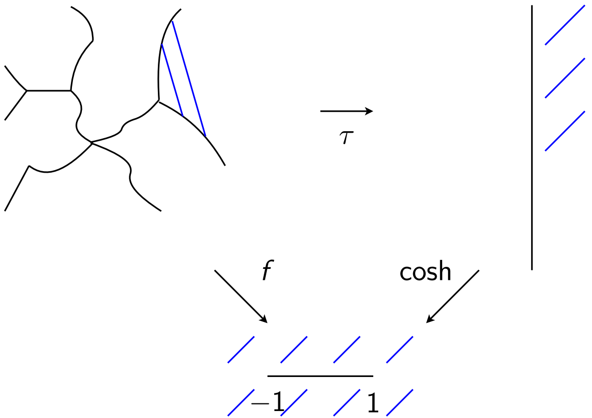

Reverse this procedure!

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Theorem (Bishop, 2011)

Suppose T is a bounded geometry tree and suppose τ is conformal from each complementary component of T to its standard version (i.e. left/right half-plane).

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Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = \sigma \circ \tau$ off $T(r_0)$ (a neighbourhood of T). The only singular values of f are ± 1 (critical values coming from the vertices of T) and the singular values assigned by the L components.

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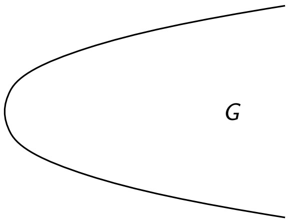
Remark

The regularity conditions are satisfied e.g. for x^{ε} if $\frac{1}{2} \leq \varepsilon < 1$.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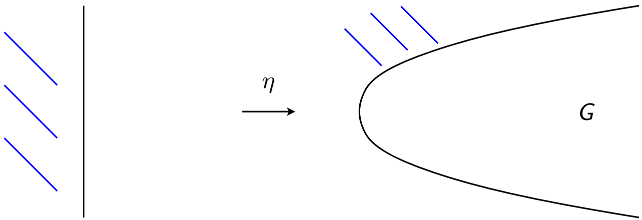
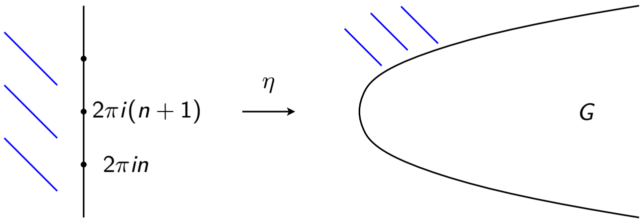
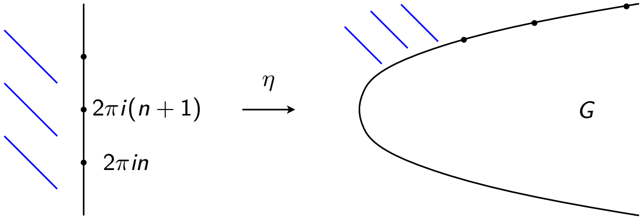

η
G



$2\pi i(n+1)$
$2\pi in$
η
G



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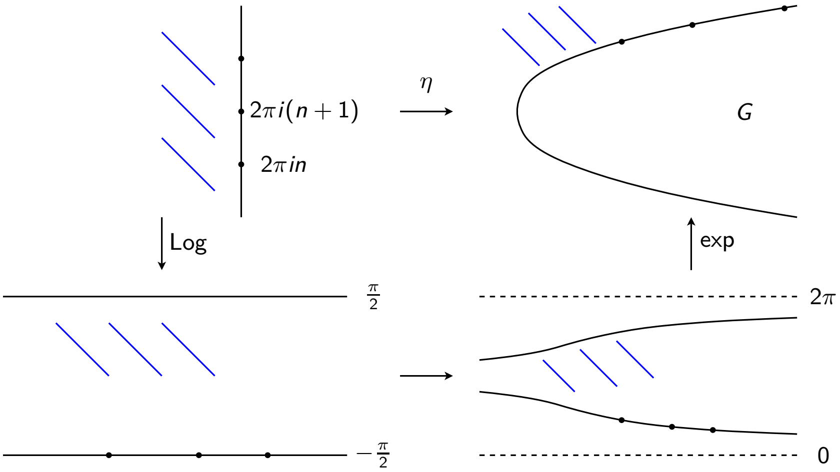

$2\pi i(n+1)$
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Log
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2π
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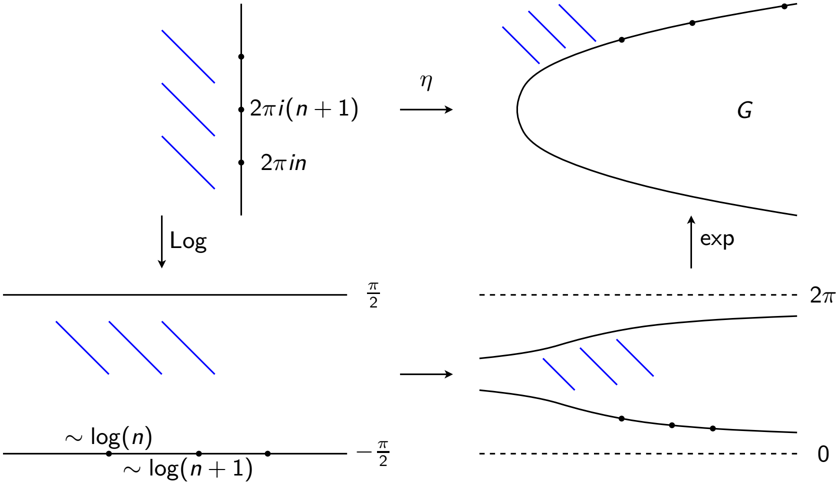
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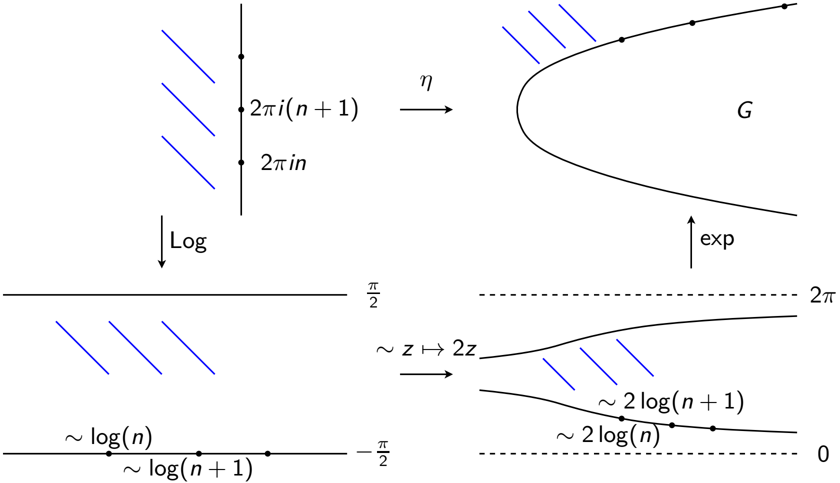

$2\pi i(n+1)$
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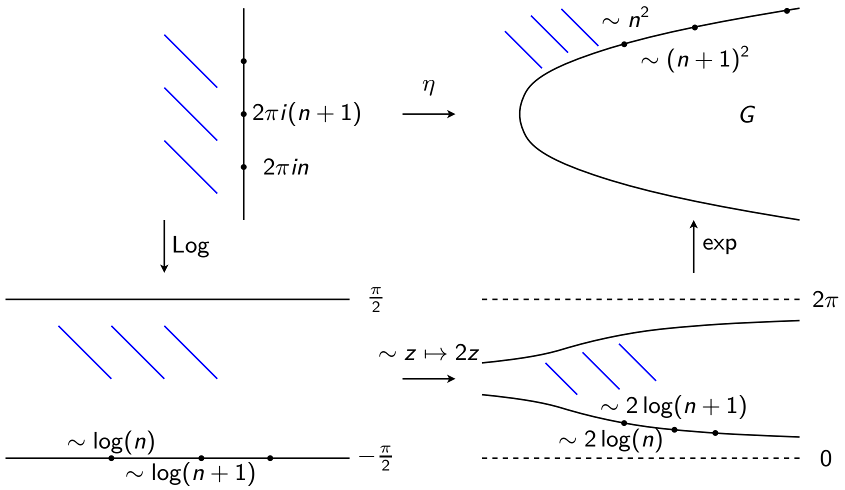

$\sim n^2$
$\sim (n+1)^2$
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- for non-adjacent edges e and f, $\frac{\text{diam}(e)}{\text{dist}(e,f)}$ is uniformly bounded: clear for edges on the same side of ∂G. For edges on opposite sides: $\text{dist}(e, f) \gtrsim \phi(n^2)$ and since $\phi(x) \geq c\sqrt{x}$ and $\ell(e) \sim n$ also in this case the quotient is bounded.

It remains to show, that there exists a conformal map $\tau : G \to \mathbb{H}_r$ such that $\ell(\tau(e)) \geq 2\pi$ for all edges e.

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$$\beta(x_2) - \alpha(x_1) \geq \frac{\pi}{2} \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\psi(t)} - 4\pi.$$

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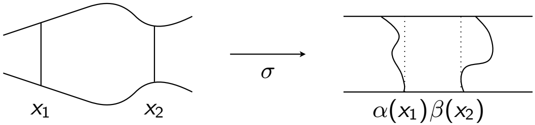
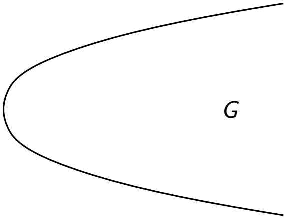
Let $\Omega = \{x + iy \ : \ |y| < \psi(x)\}$ be a horizontal strip with some non-negative funtion $\psi : \mathbb{R} \to \mathbb{R}$. Let σ map Ω conformally onto $\{x + iy \ : \ |y| < \frac{\pi}{2}\}$ such that $\mathrm{Re}(\sigma(z)) \to \pm\infty$ as $\mathrm{Re}(z) \to \pm\infty$. Let $\beta(x) = \inf_{|y|<\psi(x)} \mathrm{Re}(\sigma(x + iy))$ and $\alpha(x) = \sup_{|y|<\psi(x)} \mathrm{Re}(\sigma(x + iy))$. If $\int_{x_1}^{x_2} \frac{\mathrm{d}t}{\psi(t)} > 4$, then

$$\beta(x_2) - \alpha(x_1) \geq \frac{\pi}{2} \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\psi(t)} - 4\pi.$$

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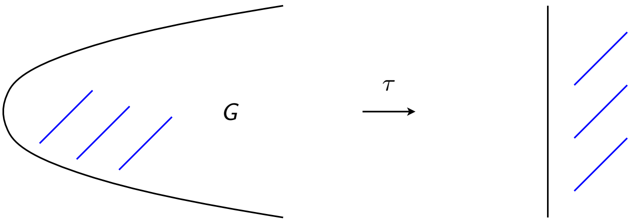
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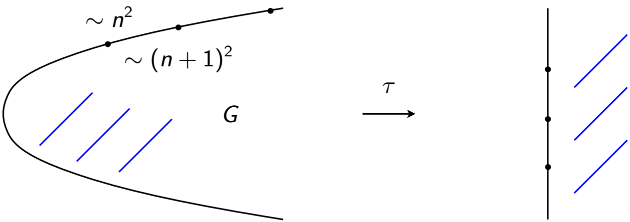
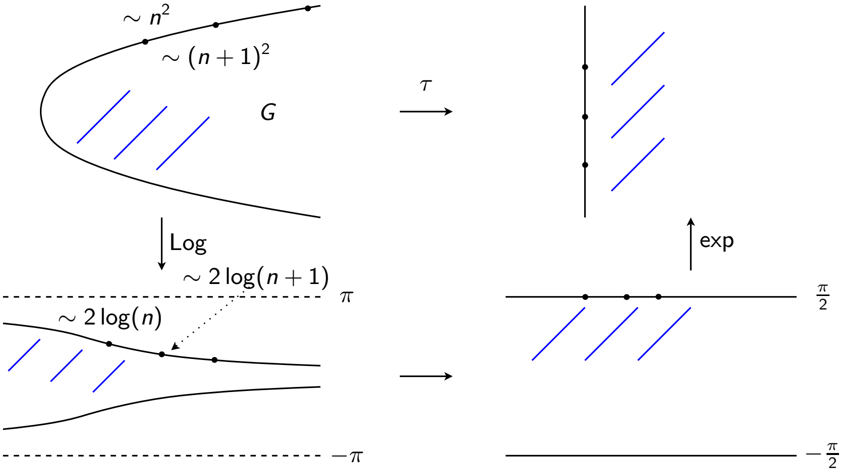

$\sim n^2$
$\sim (n+1)^2$
G
τ


$\sim n^2$
$\sim (n+1)^2$
G
τ
Log
exp
$\sim 2\log(n+1)$
π
$\sim 2\log(n)$
$-\pi$
$\frac{\pi}{2}$
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Remark

In many cases we even get $f \sim \exp \circ \omega$.

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  - construct f in class S with given Hausdorff dimension of $\mathcal{J}(f)$.
  - construct f in class S with $\dim_H(I(f)) = 1$.

Thank you very much for your attention.