Absorbing domains for meromorphic maps

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Weakly repelling fixed points

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$U \subset \mathbb{C}$ a domain, $F : U \to U$ holomorphic, such that $F^n \to \zeta$ as $n \to \infty$ for some $\zeta \in \operatorname{cl} U$ (cl denotes closure in $\overline{\mathbb{C}}$).

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We call $W \subset U$ an absorbing domain, if:

- $F(W) \subset W$ (stronger version: $F(\overline{W}) \subset W$)
- for every compact set $K \subset U$ there exists $n \geq 0$ such that $F^n(K) \subset W$.

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We say that W is converging to ζ, if $\bigcap_{n \geq 0} \operatorname{cl} F^n(W) = \{\zeta\}$.

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Examples – attracting or parabolic domains

- U basin of an attracting p-periodic point $\zeta \in U$ for a map f, $F = f^p$, W the image of a small disc around 0 under Königs or Böttcher coordinates (W simply connected)
- U basin of a parabolic p-periodic point $\zeta \in \partial U$ for a map f, $F = f^p$, W attracting petal in U (W simply connected)

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Theorem (Cowen 1981)

There exists a simply connected absorbing domain $V \subset \mathbb{H}$, converging to ∞. Moreover, there is a domain Ω equal to $\mathbb{H}$ or $\mathbb{C}$, a holomorphic map $\varphi : \mathbb{H} \to \Omega$ and a Möbius map T, such that:

- $\varphi \circ G = T \circ \varphi$ *on* $\mathbb{H}$,
- φ *is univalent on* V.

Moreover, (up to conjugation by a Möbius map), one of the following cases holds:

- $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$,
- $\Omega = \mathbb{H}$, $T(\omega) = a\omega$ *for some* $a > 1$,
- $\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$.

$$
\begin{array}{ccccc}
\varphi(V) & \subset & \Omega & \xrightarrow{\;T\;} & \Omega \\
\Big\downarrow \varphi^{-1} & & \Big\uparrow \varphi & & \Big\uparrow \varphi \\
V & \subset & \mathbb{H} & \xrightarrow{\;G\;} & \mathbb{H}
\end{array}
$$

Absorbing domains in non-simply connected regions

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$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{G} & \mathbb{H} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ U & \xrightarrow{F} & U \end{array}$$

Theorem (König 1999)

If for every closed curve $\gamma \subset U$ there exists $n > 0$ such that $F^n(\gamma)$ is contractible in U, then π is univalent in V, where $V \subset \mathbb{H}$ is the absorbing domain for G from Cowen's theorem, and

$$W = \pi(V)$$

is a simply connected absorbing domain in U for F, converging to ζ. Moreover, if f is a meromorphic map with finitely many poles and U is a periodic Baker domain of period p, then the above assumption is satisfied for $F = f^p$, so U has a simply connected absorbing domain.

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Remark

There are multiply connected regions U and maps $F : U \to U$ without simply connected absorbing domains.

$$\begin{array}{ccccc}
\varphi(V) & \subset & \Omega & \xrightarrow{\ T\ } & \Omega \\
\Big\downarrow \varphi^{-1} & & \Big\uparrow \varphi & & \Big\uparrow \varphi \\
V & \subset & \mathbb{H} & \xrightarrow{\ G\ } & \mathbb{H} \\
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W & \subset & U & \xrightarrow{\ F\ } & U
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Idea of the proof of König's Theorem

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Theorem (Marden–Pommerenke 1980)

Suppose $U \simeq \mathbb{H}/\Gamma$, *where* $\Gamma \simeq \pi_1(U)$ *infinitely generated Fuchsian group of deck transformations. For* $n \geq 0$ *let* $\theta_n : \Gamma \to \Gamma$ *be an endomorphism induced by* G^n *(i.e.* $G^n \circ \gamma = \theta_n(\gamma) \circ G^n$ *for* $\gamma \in \Gamma$*) and let* $N = \bigcup_{n=0}^{\infty} \ker \theta_n$. *We have* $N \simeq \bigcup_{n=0}^{\infty} \ker(F^n)^*$, *where* $(F^n)^*$ *induced endomorphism of the fundamental group* $\pi_1(U)$. *Then there exists a discontinuous group of Möbius maps* $\Gamma_\infty \simeq \Gamma/N$ *and a Riemann surface* R, *such that the mapping*

$$\Delta : U \to R, \qquad \Delta = \varphi \circ \pi^{-1} \ (mod\ \Gamma_\infty)$$

is well-defined for the mapping φ *from Cowen's Theorem. Moreover,*

$$\Delta \circ F = T \circ \Delta$$

for a Möbius map T.

Theorem (KB, N. Fagella, X. Jarque, B. Karpińska 2012)

Let $U \subset \mathbb{C}$ *be any hyperbolic domain,* $F : U \to U$ *holomorphic map, such that* $F^n \to \infty$ *as* $n \to \infty$. *Then there exists an absorbing domain* $W \subset U$ *converging to* ∞*, such that*

- $\overline{W} \subset U$,
- $F(\overline{W}) = \overline{F(W)} \subset W$.

Moreover, for every $z \in U$ *and every sequence* $r_n > 0$ *with* $\lim_{n\to\infty} r_n = \infty$*, the domain* W *can be chosen such that*

$$W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(F^n(z), r_n)$$

where $\mathcal{D}_U$ *denote hyperbolic discs in* U.
Furthermore, $F|_W$ *lifts under a universal covering of* W *to a univalent map and the induced endomorphism* $(F|_W)^*$ *of the fundamental group of* W *is an isomorphism.*

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$$N(z) = z - \frac{g(z)}{g'(z)},$$

where g entire map (polynomial or transcendental).

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Yes.