On boundaries of multiply connected wandering domains

Markus Baumgartner

Christian-Albrechts-Universität zu Kiel

Edinburgh, 23 May 2013

Outline

1 Introduction

Outline

1 Introduction

2 Results

Outline

1. Introduction
2. Results
3. Proof

Outline

1. Introduction
2. Results
3. Proof
4. Examples

Introduction

Definition (Wandering domain)

Let f be a rational or entire function. A Fatou component U is called *wandering domain* if $f^n(U) \cap f^m(U) = \emptyset$ for all $m < n$.

Introduction

Definition (Wandering domain)

Let f be a rational or entire function. A Fatou component U is called *wandering domain* if $f^n(U) \cap f^m(U) = \emptyset$ for all $m < n$.

Theorem (Sullivan 1982)

There are no wandering domains for rational functions.

First example of a wandering domain

The first example of a wandering domain is due to Baker. The function considered was

$$f(z) = C \cdot z^2 \prod_{j=1}^{\infty} \left(1 + \frac{z}{r_j}\right),$$

First example of a wandering domain

The first example of a wandering domain is due to Baker. The function considered was

$$f(z) = C \cdot z^2 \prod_{j=1}^{\infty} \left(1 + \frac{z}{r_j}\right),$$

where $C > 0$ is a small constant, r_1 is large and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers that satisfies the recurrence relation

$$r_{n+1} = C \cdot r_n^2 \prod_{j=1}^{n} \left(1 + \frac{r_n}{r_j}\right).$$

First example of a wandering domain

The first example of a wandering domain is due to Baker. The function considered was

$$f(z) = C \cdot z^2 \prod_{j=1}^{\infty} \left(1 + \frac{z}{r_j}\right),$$

where $C > 0$ is a small constant, r_1 is large and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers that satisfies the recurrence relation

$$r_{n+1} = C \cdot r_n^2 \prod_{j=1}^{n} \left(1 + \frac{r_n}{r_j}\right).$$

In 1963 Baker showed that f has multiply connected Fatou components U_n with $f(U_n) \subset U_{n+1}$, but the question whether the U_n are all different remained open. Those were the first known multiply connected Fatou components.

First example of a wandering domain

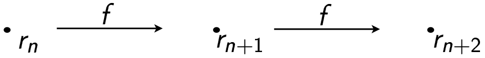
The first example of a wandering domain is due to Baker. The function considered was

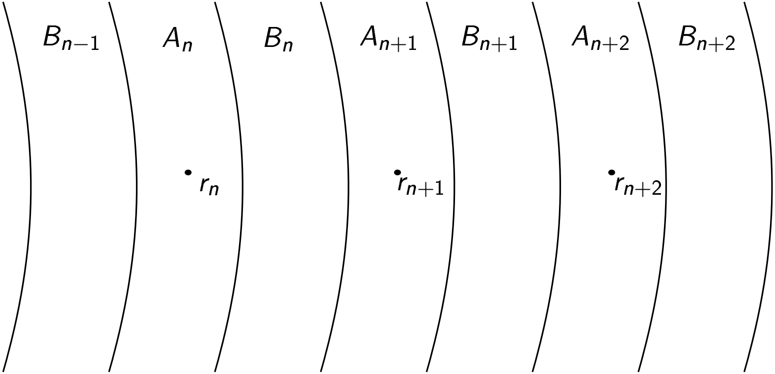
$$f(z) = C \cdot z^2 \prod_{j=1}^{\infty} \left(1 + \frac{z}{r_j}\right),$$

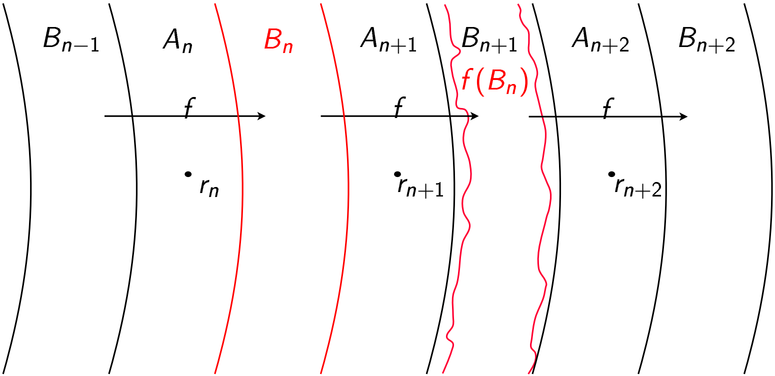
where $C > 0$ is a small constant, r_1 is large and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers that satisfies the recurrence relation

$$r_{n+1} = C \cdot r_n^2 \prod_{j=1}^{n} \left(1 + \frac{r_n}{r_j}\right).$$

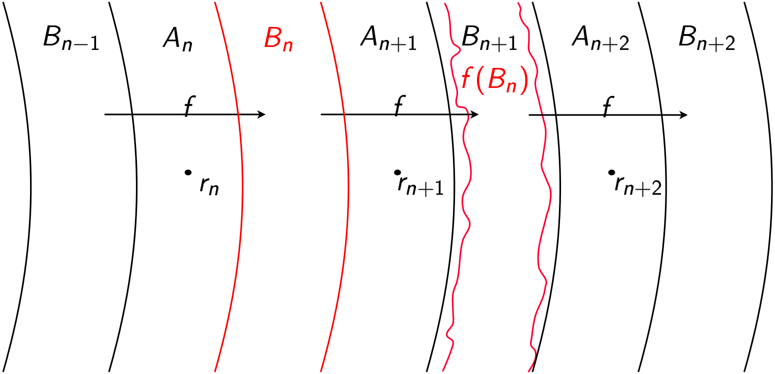
In 1963 Baker showed that f has multiply connected Fatou components U_n with $f(U_n) \subset U_{n+1}$, but the question whether the U_n are all different remained open. Those were the first known multiply connected Fatou components.
In 1976 Baker was able to show that the U_n are all different and therefore wandering domains.


r_n
f
r_{n+1}
f
r_{n+2}


B_{n-1}
A_n
B_n
A_{n+1}
B_{n+1}
A_{n+2}
B_{n+2}
r_n
r_{n+1}
r_{n+2}



- $f(B_n) \subset B_{n+1}$ (and therefore $A_{n+1} \subset f(A_n)$)



- $f(B_n) \subset B_{n+1}$ (and therefore $A_{n+1} \subset f(A_n)$)
- This implies that B_n belongs to a multiply connected wandering domain U_n.

Theorem (Baker and Dominguez 2000)

Let f be an entire function. If $\mathcal{J}(f)$ is not connected, then it is not locally connected at any point of $\mathcal{J}(f)$.

Theorem (Baker and Dominguez 2000)

Let f be an entire function. If $\mathcal{J}(f)$ is not connected, then it is not locally connected at any point of $\mathcal{J}(f)$.

This implies that $\mathcal{J}(f)$ can not be locally connected at any point for an entire function f that has a multiply connected wandering domain.

Theorem (Baker and Dominguez 2000)

Let f be an entire function. If $\mathcal{J}(f)$ is not connected, then it is not locally connected at any point of $\mathcal{J}(f)$.

This implies that $\mathcal{J}(f)$ can not be locally connected at any point for an entire function f that has a multiply connected wandering domain.

Question

Are at least the different components of $\mathcal{J}(f)$ locally connected?

Theorem (Baker and Dominguez 2000)

Let f be an entire function. If $\mathcal{J}(f)$ is not connected, then it is not locally connected at any point of $\mathcal{J}(f)$.

This implies that $\mathcal{J}(f)$ can not be locally connected at any point for an entire function f that has a multiply connected wandering domain.

Question

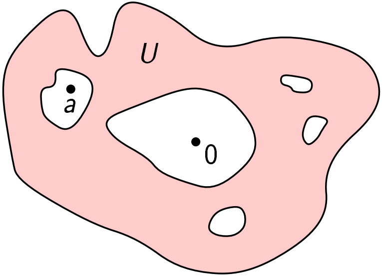
Are at least the different components of $\mathcal{J}(f)$ locally connected?

We want to show that under suitable conditions every boundary component of a multiply connected wandering domain is a curve or even a Jordan curve and therefore locally connected.

Results

Definition (Inner and outer boundary)

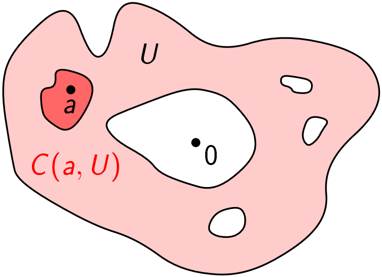
Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.



Results

Definition (Inner and outer boundary)

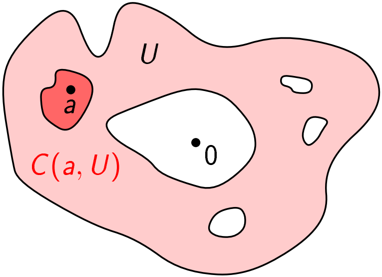
Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.



Results

Definition (Inner and outer boundary)

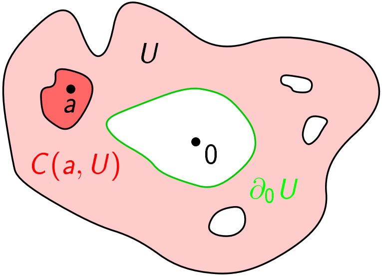
Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.
For $0 \notin U$ we define $\partial_0 U = \partial C(0, U)$ as *inner boundary component* and $\partial_\infty U = \partial C(\infty, U)$ as *outer boundary component*. We call $\partial_0 U$ and $\partial_\infty U$ *big boundary components*.



Results

Definition (Inner and outer boundary)

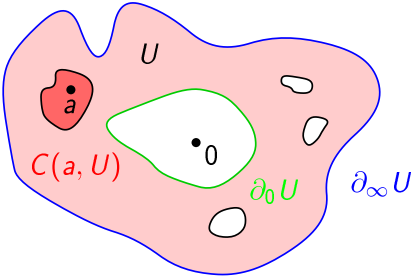
Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.
For $0 \notin U$ we define $\partial_0 U = \partial C(0, U)$ as *inner boundary component* and $\partial_\infty U = \partial C(\infty, U)$ as *outer boundary component*. We call $\partial_0 U$ and $\partial_\infty U$ *big boundary components*.



Results

Definition (Inner and outer boundary)

Let $U \subset \mathbb{C}$ be a domain and let $a \in \overline{\mathbb{C}} \setminus U$. We denote by $C(a, U)$ the component of $\overline{\mathbb{C}} \setminus U$ that contains a.
For $0 \notin U$ we define $\partial_0 U = \partial C(0, U)$ as *inner boundary component* and $\partial_\infty U = \partial C(\infty, U)$ as *outer boundary component*. We call $\partial_0 U$ and $\partial_\infty U$ *big boundary components*.



Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.

Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.
Suppose that there exists a sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ as well as $\alpha, \beta > 0$ such that for a sequence of annuli $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$ the following conditions hold:

Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.
Suppose that there exists a sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ as well as $\alpha, \beta > 0$ such that for a sequence of annuli $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$ the following conditions hold:

- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$

Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.
Suppose that there exists a sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ as well as $\alpha, \beta > 0$ such that for a sequence of annuli $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$ the following conditions hold:

- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$
- $C_{n+1} \subset f(C_n)$

Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.
Suppose that there exists a sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ as well as $\alpha, \beta > 0$ such that for a sequence of annuli $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$ the following conditions hold:

- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$
- $C_{n+1} \subset f(C_n)$
- *There exists $m > \frac{\alpha}{\beta}$ such that for all $z \in f^{-1}(C_{n+1}) \cap C_n$*

$$\left|\frac{z \cdot f'(z)}{f(z)}\right| \geq m \quad \text{and} \quad \text{Re}\left(\frac{z \cdot f'(z)}{f(z)}\right) > 0.$$

Theorem

Let f be an entire function with a multiply connected wandering domain $U = U_0$. Denote $U_n = f^n(U)$.
Suppose that there exists a sequence of positive real numbers $(r_n)_{n\in\mathbb{N}}$ as well as $\alpha, \beta > 0$ such that for a sequence of annuli $C_n := \{z \in \mathbb{C} : \alpha r_n \le |z| \le \beta r_n\}$ the following conditions hold:

- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$
- $C_{n+1} \subset f(C_n)$
- *There exists $m > \frac{\alpha}{\beta}$ such that for all $z \in f^{-1}(C_{n+1}) \cap C_n$*

$$\left|\frac{z \cdot f'(z)}{f(z)}\right| \ge m \quad \text{and} \quad \operatorname{Re}\left(\frac{z \cdot f'(z)}{f(z)}\right) > 0.$$

Then all wandering domains, which belong to the orbit of U_n, are bounded by a countable number of closed curves.

The proof of the theorem shows that we additionally have:

The proof of the theorem shows that we additionally have:

- $\partial_\infty U_{n-1} = \partial_0 U_n$

The proof of the theorem shows that we additionally have:

- $\partial_\infty U_{n-1} = \partial_0 U_n$
- $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are Jordan curves.

The proof of the theorem shows that we additionally have:

- $\partial_\infty U_{n-1} = \partial_0 U_n$
- $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are Jordan curves.
- If Z is a boundary component of a wandering domain, which belongs to the orbit of U_n, such that $f^j(Z)$ does not contain any critical points for all $j \in \mathbb{N}_0$, then Z is a Jordan curve.

The proof of the theorem shows that we additionally have:

- $\partial_\infty U_{n-1} = \partial_0 U_n$
- $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are Jordan curves.
- If Z is a boundary component of a wandering domain, which belongs to the orbit of U_n, such that $f^j(Z)$ does not contain any critical points for all $j \in \mathbb{N}_0$, then Z is a Jordan curve.

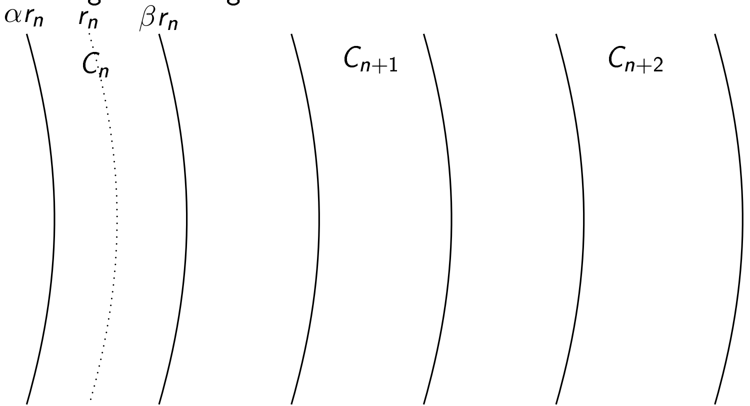
The theorem works for Baker's first example of a wandering domain.

Proof

Understanding the setting of the theorem:

Proof

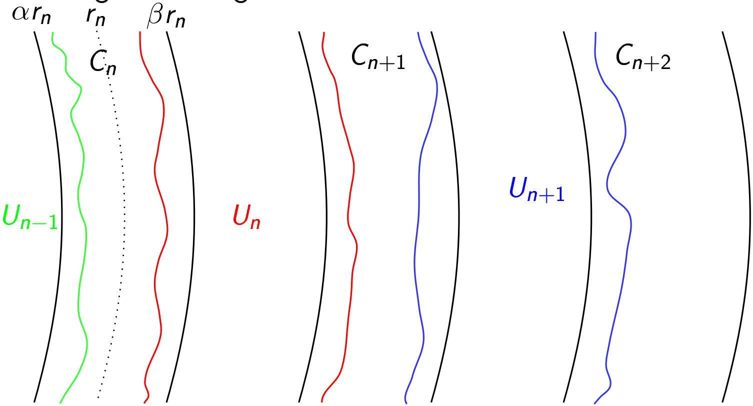
Understanding the setting of the theorem:



- $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$

Proof

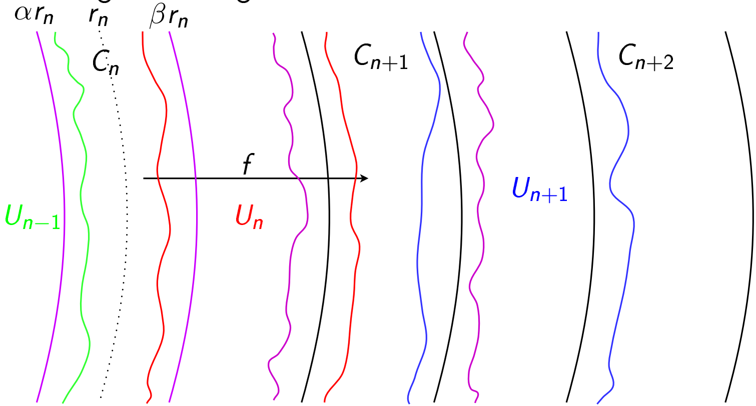
Understanding the setting of the theorem:



- $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$
- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$

Proof

Understanding the setting of the theorem:



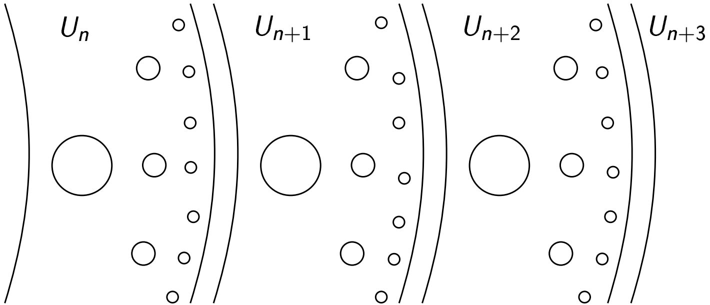
- $C_n := \{z \in \mathbb{C} : \alpha r_n \leq |z| \leq \beta r_n\}$
- $\partial_0 C_n \subset U_{n-1}$, $\partial_\infty C_n \subset U_n$
- $C_{n+1} \subset f(C_n)$

Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.

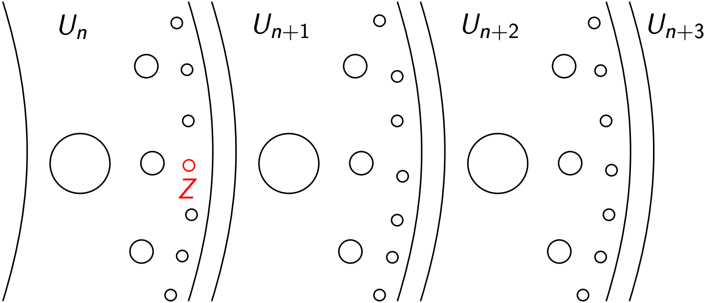
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



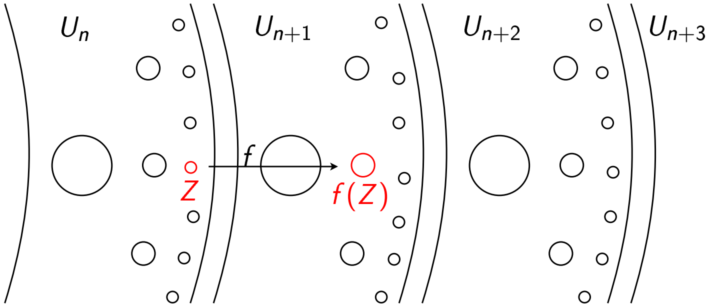
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



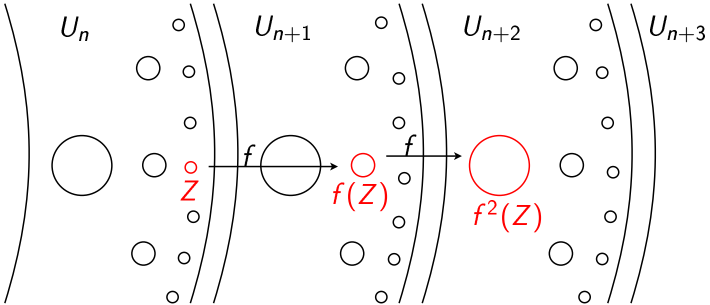
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



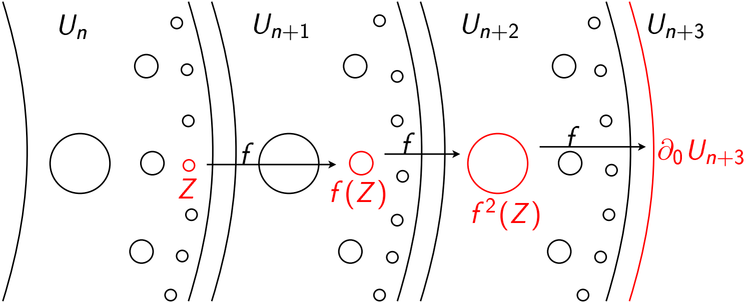
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



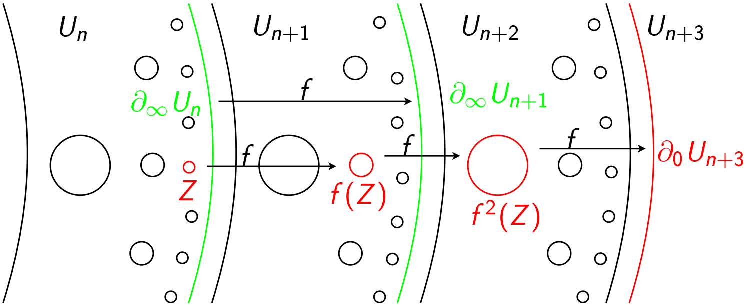
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



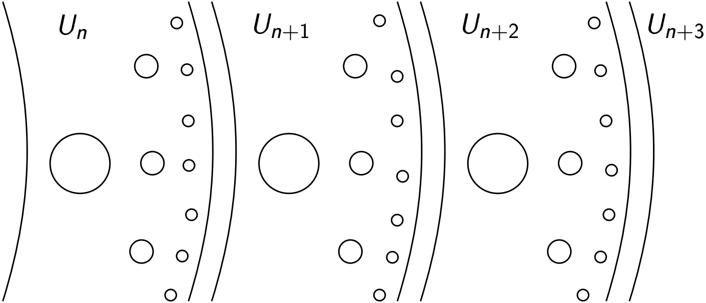
Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



Lemma

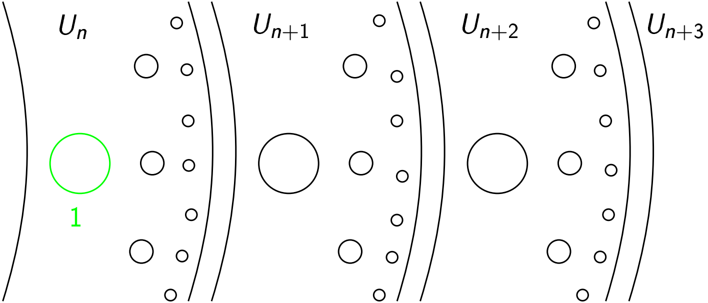
Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



For this reason it makes sense to group the boundary components in certain 'levels' indicated by the iterations they need to be mapped onto a big boundary component.

Lemma

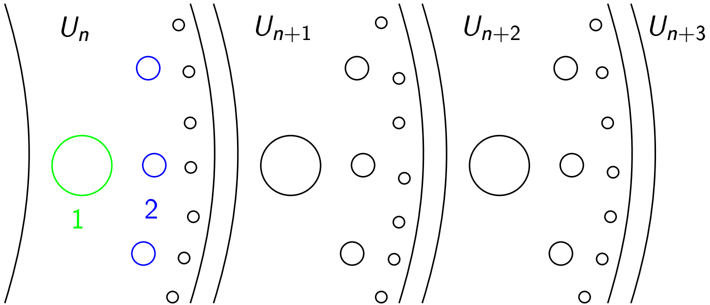
Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



For this reason it makes sense to group the boundary components in certain 'levels' indicated by the iterations they need to be mapped onto a big boundary component.

Lemma

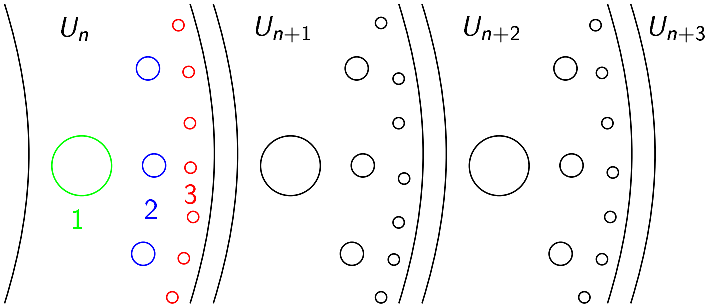
Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



For this reason it makes sense to group the boundary components in certain 'levels' indicated by the iterations they need to be mapped onto a big boundary component.

Lemma

Let Z be a boundary component of U.
If $Z \neq \partial_\infty U$ there exists $q \in \mathbb{N}_0$ such that $f^j(Z) = \partial_0 U_j$ for all $j \geq q$.
If $Z = \partial_\infty U$ we have $f^j(Z) = \partial_\infty U_j$ for all $j \in \mathbb{N}_0$.



For this reason it makes sense to group the boundary components in certain 'levels' indicated by the iterations they need to be mapped onto a big boundary component.

Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves.

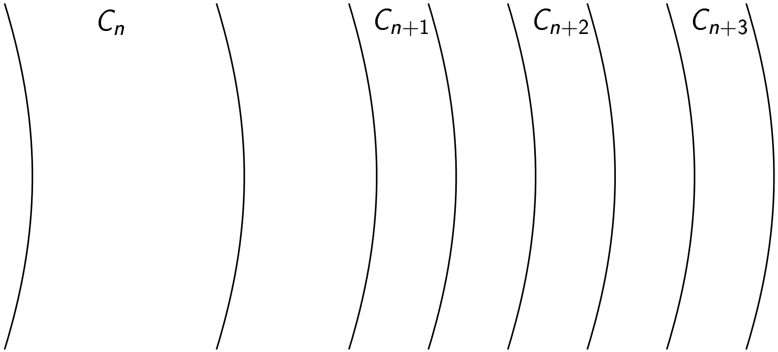
Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.

Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.
Define for all $k \in \mathbb{N}$

$$\Gamma_k := \{z \in C_n : f^j(z) \in C_{n+j} \text{ for all j=1,...,k}\}.$$

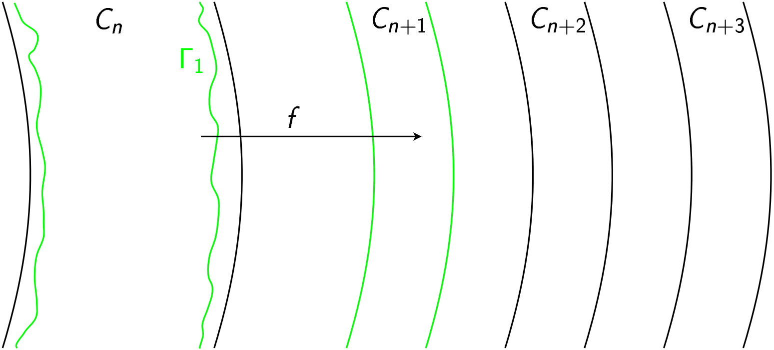
Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.
Define for all $k \in \mathbb{N}$

$$\Gamma_k := \{z \in C_n : f^j(z) \in C_{n+j} \text{ for all j=1,...,k}\}.$$



Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.
Define for all $k \in \mathbb{N}$

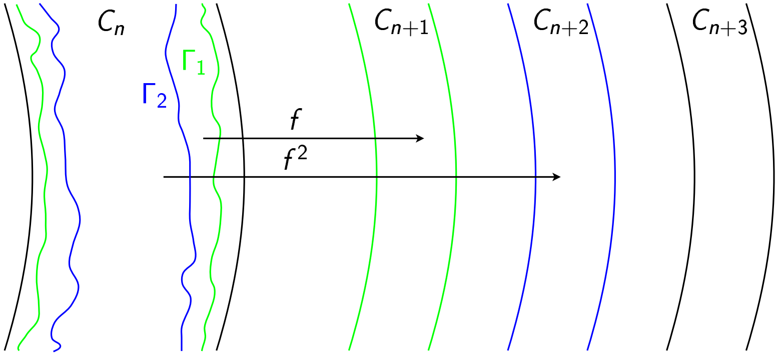
$$\Gamma_k := \{z \in C_n : f^j(z) \in C_{n+j} \text{ for all j=1,...,k}\}.$$



Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.

Define for all $k \in \mathbb{N}$

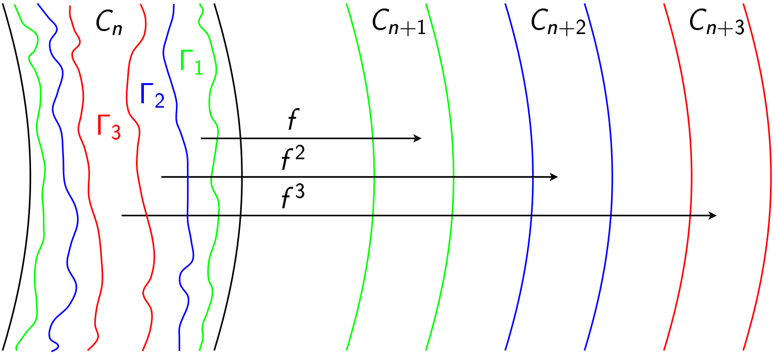
$$\Gamma_k := \{z \in C_n : f^j(z) \in C_{n+j} \text{ for all j=1,...,k}\}.$$



Since every boundary component is eventually mapped onto a big boundary component by the last lemma it suffices to show that they are curves. Therefore we want to show that $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are both curves that coincide.

Define for all $k \in \mathbb{N}$

$$\Gamma_k := \{z \in C_n : f^j(z) \in C_{n+j} \text{ for all j=1,...,k}\}.$$



The inequality $\left|\frac{z\cdot f'(z)}{f(z)}\right| \geq m$ implies that there are no critical points inside the Γ_k. So all Γ_k are topological annuli by the Riemann-Hurwitz-formula that are bounded by Jordan curves.

The inequality $\left|\frac{z \cdot f'(z)}{f(z)}\right| \geq m$ implies that there are no critical points inside the Γ_k. So all Γ_k are topological annuli by the Riemann-Hurwitz-formula that are bounded by Jordan curves.
The inequality also implies the following lemma:

The inequality $\left|\frac{z \cdot f'(z)}{f(z)}\right| \geq m$ implies that there are no critical points inside the Γ_k. So all Γ_k are topological annuli by the Riemann-Hurwitz-formula that are bounded by Jordan curves.
The inequality also implies the following lemma:

Lemma

There exists $\varrho > 1$ such that for all $k \in \mathbb{N}$ and $z \in \Gamma_k$

$$|(f^k)'(z)| \geq \varrho^k \cdot \frac{r_{n+k}}{r_n}$$

The inequality $\left|\frac{z \cdot f'(z)}{f(z)}\right| \geq m$ implies that there are no critical points inside the Γ_k. So all Γ_k are topological annuli by the Riemann-Hurwitz-formula that are bounded by Jordan curves.
The inequality also implies the following lemma:

Lemma

There exists $\varrho > 1$ *such that for all* $k \in \mathbb{N}$ *and* $z \in \Gamma_k$

$$|(f^k)'(z)| \geq \varrho^k \cdot \frac{r_{n+k}}{r_n}$$

Therefore f^k is expanding inside Γ_k and this implies that $f^{-k} : C_{n+k} \to \Gamma_k$ is contracting.

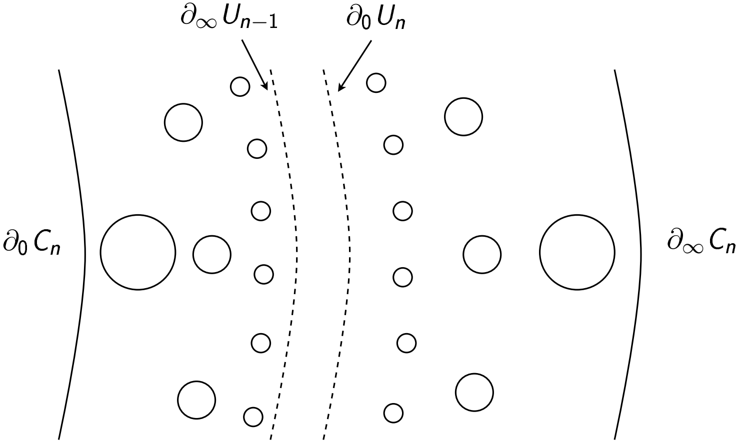
The inequality $\left|\frac{z \cdot f'(z)}{f(z)}\right| \geq m$ implies that there are no critical points inside the Γ_k. So all Γ_k are topological annuli by the Riemann-Hurwitz-formula that are bounded by Jordan curves.
The inequality also implies the following lemma:

Lemma

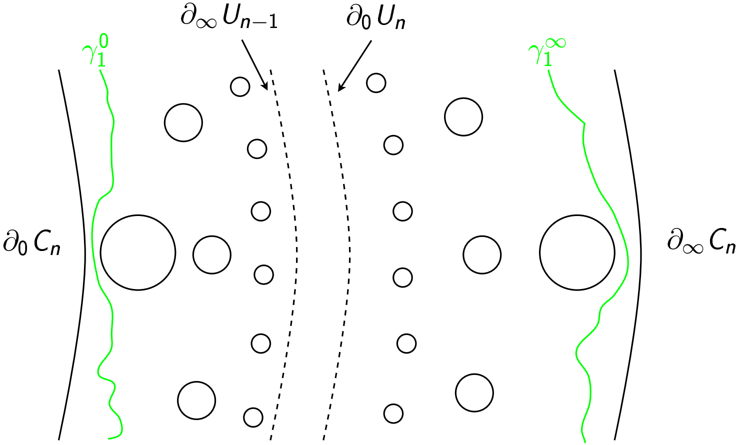
There exists $\varrho > 1$ such that for all $k \in \mathbb{N}$ and $z \in \Gamma_k$

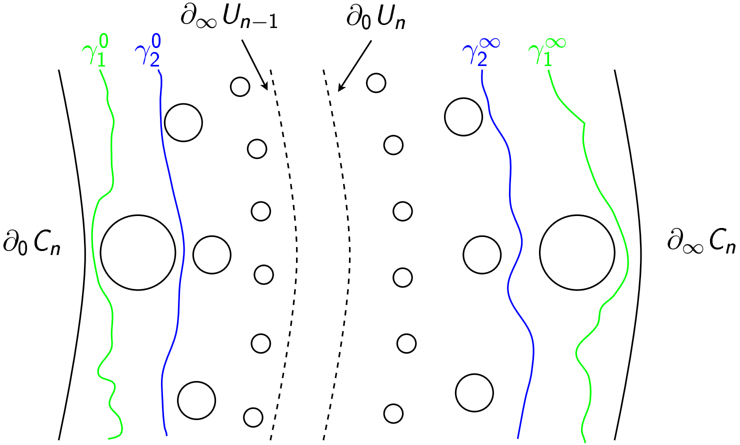
$$|(f^k)'(z)| \geq \varrho^k \cdot \frac{r_{n+k}}{r_n}$$

Therefore f^k is expanding inside Γ_k and this implies that $f^{-k} : C_{n+k} \to \Gamma_k$ is contracting.
We parametrise now $\partial_0 \Gamma_k$ and $\partial_\infty \Gamma_k$ as curves by γ_k^0 and γ_k^∞ respectively.
Thereby one has to check that the parametrisations are compatible with each other. Here $\mathrm{Re}\left(\frac{z \cdot f'(z)}{f(z)}\right) > 0$ is used. It ensures that the curves are not distorted too much under iteration.

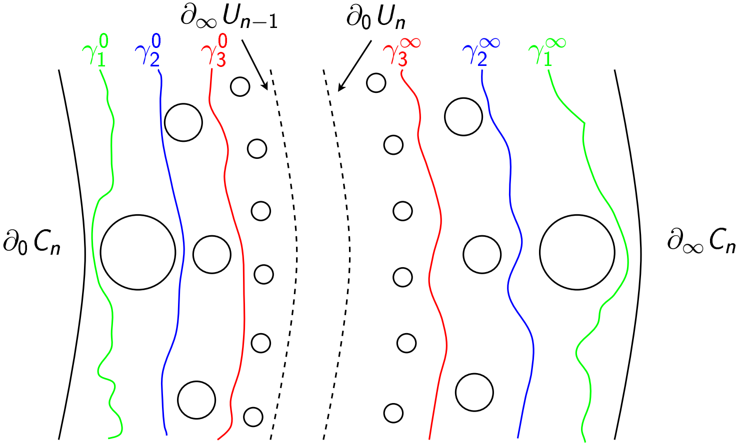


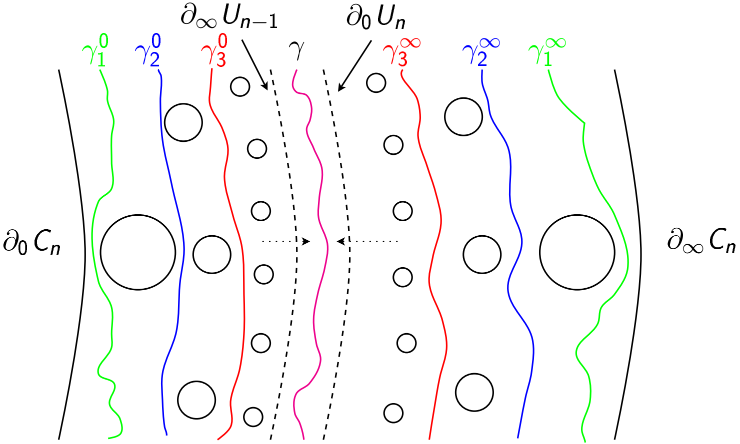
$\partial_\infty U_{n-1}$
$\partial_0 U_n$
$\partial_0 C_n$
$\partial_\infty C_n$


$\partial_\infty U_{n-1}$
$\partial_0 U_n$
γ_1^0
γ_1^∞
$\partial_0 C_n$
$\partial_\infty C_n$



$\partial_\infty U_{n-1}$
$\partial_0 U_n$
γ_1^0
γ_2^0
γ_2^∞
γ_1^∞
$\partial_0 C_n$
$\partial_\infty C_n$


$\partial_\infty U_{n-1}$
$\partial_0 U_n$
γ_1^0
γ_2^0
γ_3^0
γ_3^∞
γ_2^∞
γ_1^∞
$\partial_0 C_n$
$\partial_\infty C_n$



Then we use that f^{-k} is contracting to show that the curves γ_k^0 and γ_k^∞ converge uniformly to the same curve γ with

$$\operatorname{trace}(\gamma) = \bigcap_{k\in\mathbb{N}} \Gamma_k .$$

By positioning of C_n to U_{n-1} and U_n we have

$$\partial_\infty U_{n-1} = \mathrm{trace}(\gamma) = \partial_0 U_n.$$

By positioning of C_n to U_{n-1} and U_n we have

$$\partial_\infty U_{n-1} = \text{trace}(\gamma) = \partial_0 U_n.$$

Now since $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are curves and therefore locally connected every point on trace(γ) is accessible in U_{n-1} and in U_n.

By positioning of C_n to U_{n-1} and U_n we have

$$\partial_\infty U_{n-1} = \text{trace}(\gamma) = \partial_0 U_n.$$

Now since $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are curves and therefore locally connected every point on trace(γ) is accessible in U_{n-1} and in U_n.
Thus a theorem of Schönflies yields that γ is in fact a Jordan curve.

By positioning of C_n to U_{n-1} and U_n we have

$$\partial_\infty U_{n-1} = \mathrm{trace}(\gamma) = \partial_0 U_n.$$

Now since $\partial_\infty U_{n-1}$ and $\partial_0 U_n$ are curves and therefore locally connected every point on trace(γ) is accessible in U_{n-1} and in U_n.
Thus a theorem of Schönflies yields that γ is in fact a Jordan curve.
As mentioned before every boundary component of U will be eventually mapped onto such a big boundary component, so every boundary component of U is either a curve or even a Jordan curve if there are no critical points in its forward orbit.

Examples

In the following we are looking at three different examples of entire functions with multiply connected wandering domains to which we can apply the theorem.

Examples

In the following we are looking at three different examples of entire functions with multiply connected wandering domains to which we can apply the theorem.

Bergweiler's and Zheng's example

$$f(z) = C \cdot z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right),$$

where $C > 0$, $k \in \mathbb{N}$ and $(a_j)_{j\in\mathbb{N}}$ is a complex sequence with $|a_j| = r_j$ and $(r_j)_{j\in\mathbb{N}}$ is a fast growing sequence of positive real numbers.

Examples

In the following we are looking at three different examples of entire functions with multiply connected wandering domains to which we can apply the theorem.

Bergweiler's and Zheng's example

$$f(z) = C \cdot z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right),$$

where $C > 0$, $k \in \mathbb{N}$ and $(a_j)_{j\in\mathbb{N}}$ is a complex sequence with $|a_j| = r_j$ and $(r_j)_{j\in\mathbb{N}}$ is a fast growing sequence of positive real numbers.

This example includes the first example of Baker.

Baker's infinite connectivity example

$$f(z) = C \cdot \prod_{j=1}^{\infty} \left(1 - \frac{z}{r_j}\right)^k ,$$

where $C > 0$, $k \in \mathbb{N}$ and $(r_j)_{j \in \mathbb{N}}$ is a fast growing sequence of positive real numbers.

Baker's infinite connectivity example

$$f(z) = C \cdot \prod_{j=1}^{\infty} \left(1 - \frac{z}{r_j}\right)^k ,$$

where $C > 0$, $k \in \mathbb{N}$ and $(r_j)_{j \in \mathbb{N}}$ is a fast growing sequence of positive real numbers.

This example includes the first example of Baker (1984) with a wandering domain with infinite connectivity.

Baker's infinite connectivity example

$$f(z) = C \cdot \prod_{j=1}^{\infty} \left(1 - \frac{z}{r_j}\right)^k ,$$

where $C > 0$, $k \in \mathbb{N}$ and $(r_j)_{j \in \mathbb{N}}$ is a fast growing sequence of positive real numbers.

This example includes the first example of Baker (1984) with a wandering domain with infinite connectivity.
Bergweiler and Zheng showed that Baker's first example of a wandering domain has also infinite connectivity.

The following example is due to Bishop (2011). It was the initial example for my work.

The following example is due to Bishop (2011). It was the initial example for my work.

Bishop's example

$$f(z) = F_0(z) \cdot \prod_{j=1}^{\infty} \left(1 - \frac{1}{2} \left(\frac{z}{r_j} \right)^{k_j} \right),$$

where $F_0(z)$ is a certain polynomial and $(r_j)_{j \in \mathbb{N}}$ and $(k_j)_{j \in \mathbb{N}}$ are fast growing sequences of positive real numbers.

The following example is due to Bishop (2011). It was the initial example for my work.

Bishop's example

$$f(z) = F_0(z) \cdot \prod_{j=1}^{\infty} \left(1 - \frac{1}{2} \left(\frac{z}{r_j} \right)^{k_j} \right),$$

where $F_0(z)$ is a certain polynomial and $(r_j)_{j\in\mathbb{N}}$ and $(k_j)_{j\in\mathbb{N}}$ are fast growing sequences of positive real numbers.

Bishop showed that the Julia set of this function has Hausdorff dimension 1 and that the Fatou set consists of multiply connected wandering domains which are bounded by $C^1 - curves$.

The following example is due to Bishop (2011). It was the initial example for my work.

Bishop's example

$$f(z) = F_0(z) \cdot \prod_{j=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{r_j}\right)^{k_j}\right),$$

where $F_0(z)$ is a certain polynomial and $(r_j)_{j\in\mathbb{N}}$ and $(k_j)_{j\in\mathbb{N}}$ are fast growing sequences of positive real numbers.

Bishop showed that the Julia set of this function has Hausdorff dimension 1 and that the Fatou set consists of multiply connected wandering domains which are bounded by $C^1 - curves$.
Our proofs use some of Bishop's ideas. But the arguments to show that the boundaries are $C^1 - curves$ do not work for the other examples.

Thank you for your attention.