Hyperbolic geometry

Alan F. Beardon

"There is pleasure in recognising old things from a new viewpoint"
(Richard Feynman)

20 May 2013

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- Normalisation is the enemy of invariance.

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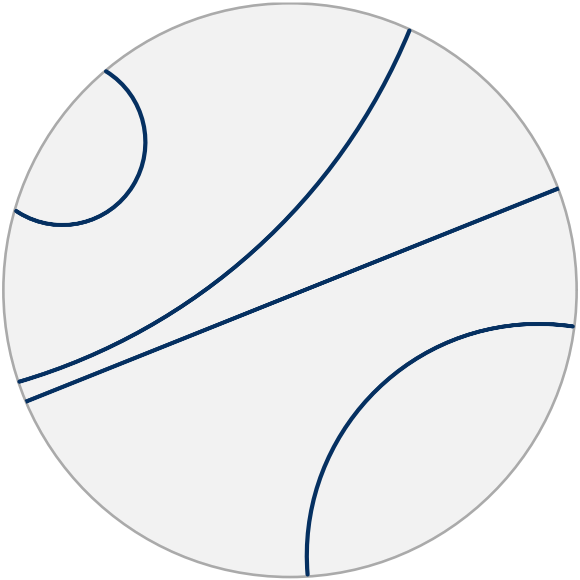
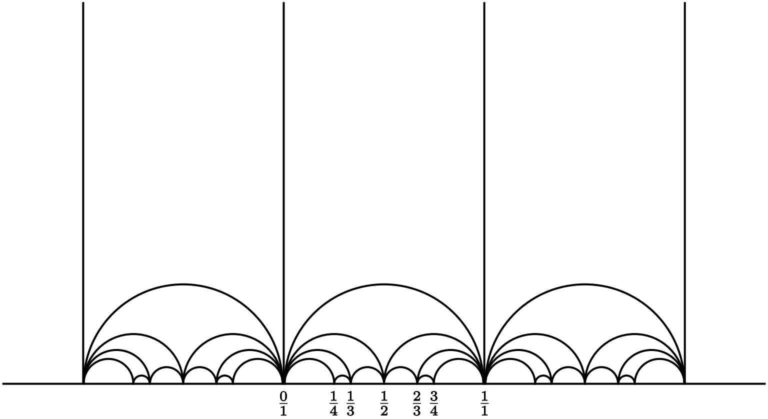
- Angles are measured as in Euclidean geometry.

Geodesics in the hyperbolic plane $\{z : |z| < 1\}$



Geodesics in the hyperbolic plane $\{x + iy : y > 0\}$



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- The hyperbolic isometries are the Möbius maps $\mathbb{D} \to \mathbb{D}$.

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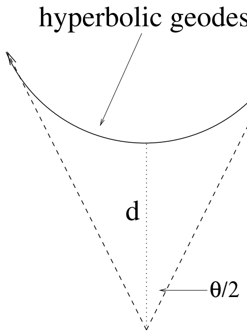
- **Hyperbolic geometry:** if a point x is at a distance d away from a geodesic, the geodesic subtends an angle θ at x, where $\sinh d \tan(\theta/2) = 1$. Here d is relevant because there are no similarities in hyperbolic geometry.
  Note that $\theta \to 0 \Leftrightarrow d \to +\infty$; $\quad d \sim \log(1/\theta), \quad \theta \sim 4/e^d$.



Simply connected regions

Let Ω be a simply connected region that is conformally equivalent to $\mathbb{D}$. Then the hyperbolic metric of Ω is defined so as to make it conformally invariant. Thus if $f : \Omega \to \mathbb{D}$ is a conformal map, then the hyperbolic metric of Ω satisfies

$$\lambda_\Omega(z)\,|dz| = \lambda_{\mathbb{D}}\big(f(z)\big)\,|df(z)|,$$

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We easily find that (for simply connected regions)

$$\frac{1}{2\mathrm{dist}(z,\Omega)} \le \lambda_\Omega(z) \le \frac{2}{\mathrm{dist}(z,\Omega)},$$

where

$$\mathrm{dist}(z,\Omega) = \inf_{w\in\partial\Omega} |z-w|.$$

Functions and metrics

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If we state our theorems in terms of metrics (instead of functions), then the results can be investigated for other metrics too, e.g. the Hilbert metric, the quasiconformal metric,... .

Caratheòdory's Kernel Theorem

Given hyperbolic regions D_1 and D_2, *any* conformal map f from D_1 onto D_2 transfers the hyperbolic metric $\lambda_1(z)\,|dz|$ to a hyperbolic metric $\lambda_2(z)\,|dz|$ that is *independent of* f.

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- Suppose that $D_1, D_2, \ldots$ is a sequence of regions, each with a hyperbolic metric $\lambda_{D_n}(z)\,|dz|$.
- Suppose that $D_n \to D$ (in a precise sense - which is omitted here)
- even though the D_n are connected, D may have many components.

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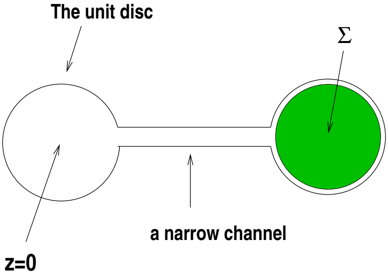
- Suppose that $D_n \to D$ (in a precise sense - which is omitted here)
- even though the D_n are connected, D may have many components.

Then $\lambda_n(z)\,|dz|$ converges on each component of D to the hyperbolic metric of that component.

This theorem is about metrics (NOT mappings),
and it holds for other metrics (eg the quasi-hyperbolic metric).

Conformal images



Consider the conformal map of the region D onto $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. Where does $f(\Sigma)$ lie in $\mathbb{D}$?

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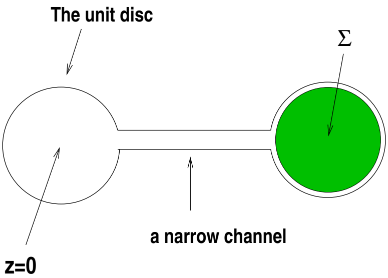

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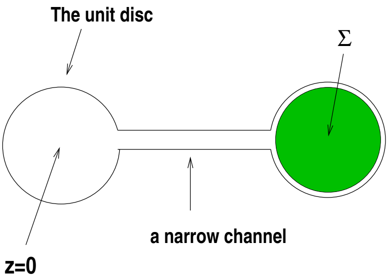

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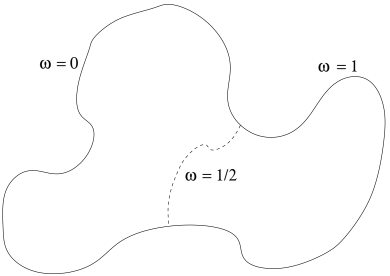
- Thus $f(\Sigma)$ is far from the origin, and so lies in some small annulus $\{z : 1 - \delta < |z| < 1\}$.
- By the 'Angle of Parallelism, $f(\Sigma)$ lies in some small Euclidean neighbourhood of some ζ with $|\zeta| = 1$. (In fact, $\zeta = 1$).

Geodesics

Difficult problem: obtain useful information about the location of geodesics in a plane region.

Given a region bounded by a Jordan curve J, let w_1 and w_2 be two points of J, and let $\omega(z)$ be the harmonic measure of one of the arcs of J with endpoints w_j. Then the geodesic joining w_1 to w_2 is the level curve $\omega(z) = \frac{1}{2}$.



Theorem (The Schwarz-Pick Lemma)

(1) Let D_1 and D_2 be simply connected regions;
(2) let $f : D_1 \to D_2$ be holomorphic;
(3) let D_j have hyperbolic distance ρ_j, $j = 1, 2$.
Then

$$\rho_2\big(f(z), f(w)\big) \leq \rho_1(z, w).$$

Equality for some z and $w \Rightarrow f$ is a conformal map of D_1 onto D_2.

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There is also a version for multiply connected regions.

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For z and w in $\mathbb{D}$, we write

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Theorem (Beardon-Minda)

For all w in $\mathbb{D}$, and f not an isometry, $z \mapsto f^*(z, w)$ is a holomorphic map of $\mathbb{D}$ into itself, so

$$\rho\big(f^*(z_1, w), f^*(z_2, w)\big) \leq \rho(z_1, z_2)$$

This result contains almost all (or perhaps all) of the many variations on the Schwarz-Pick Lemma.

The Denjoy-Wolff Theorem

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If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then there exists ζ in the closed unit disc $\overline{\mathbb{D}}$ such that $f^n(z) \to \zeta$, uniformly on compact subsets of $\mathbb{D}$, where f^n is the n-th iterate of f.

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We recall Banach's fixed point theorem: if f is a map of a complete metric space (X, d) into itself, and for some k in $(0, 1)$ we have

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If $d\big(f(x), f(y)\big) < d(x, y)$, take the one point compactification $X \cup \{\infty\}$ of X and ask for the conclusion that either $f^n(x)$ converges to a fixed point of f in X, or $f^n(x) \to \infty$.

There exist versions of the Denjoy-Wolff theorem in the following circumstances:

- *distance-decreasing* maps of $\mathbb{D}$ into itself (which need not be holomorphic);
- distance-decreasing maps of (suitable) manifolds of variable negative curvature, in all dimensions;
- distance-decreasing maps of strictly convex regions in $\mathbb{R}^n$ with the Hilbert metric (a substitute for the hyperbolic metric).

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In a similar way, Julia's lemma and the ideas about angular derivatives do not depend on the maps being holomorphic.

Multiply connected regions

The three classical geometries of constant curvature are
the hyperbolic plane (curvature -1);
the Euclidean plane (curvature 0);
the spherical plane (or Riemann sphere) (curvature $+1$).

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Examples: a torus is $\mathbb{C}/G$, where $G = \langle z+1, z+i \rangle$;
an annulus $\mathcal{A}$ is $\mathbb{H}/G$, where $G = \langle 2z \rangle$.

Normal families

A family $\mathcal{F}$ of functions meromorphic in a domain D is *normal in* D if, for every $f_1, f_2, \ldots$ chosen from $\mathcal{F}$, there is a subsequence f_{n_j} that converges to some f, uniformly on compact subsets.

Given a rational map R, let $\mathcal{R}$ be the family of iterates of R. the *Fatou set* is the set of points which have a neighbourhood in which $\mathcal{R}$ is a normal family; the *Julia set* is its complement.

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Let a, b and c be distinct points in $\mathbb{C}_\infty$, and let D be a domain. The family of maps $f : D \to \mathbb{C}_\infty \backslash \{a, b, c\}$ is normal in D.

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Because a region carries a hyperbolic metric if and only if its complement has at least three points (so why avoid this?).

The proof of Montel's Theorem

- The space $\mathbb{C}_\infty \backslash \{0, 1, \infty\}$ with its hyperbolic metric is such that the distance from any point to any one of the punctures is infinite.
- Now consider holomorphic maps $f_n : \mathbb{D} \to \mathbb{C}_\infty \backslash \{0, 1, \infty\}$, $n = 1, 2, \ldots$.
- For each positive integer m let $\Delta_m = \{z \in \mathbb{D} : \rho(z, 0) < m\}$, and fix the integer m.
- Construct small neighbourhoods $\mathcal{N}_0$ of 0, $\mathcal{N}_1$ of 1, and $\mathcal{N}_\infty$ of ∞, each a distance more than $2m$ apart from the other two.
- For each n, $f_n(\Delta_m)$ has diameter at most $2m$ (Schwarz-Pick), so it can meet at most one of three neighbourhoods.
- Thus there is a subsequence f_{n_j} such that $f_{n_j}(\Delta_m)$ is disjoint from some open set $\mathcal{N}_w$, for some w in $\{0, 1, \infty\}$. Thus (by elementary arguments) $\{f_{n_1}, f_{n_2}, \ldots\}$ is normal in Δ_m.
- Thus $\{f_1, f_2, \ldots\}$ is normal in Δ_m, and hence in $\cup_m \Delta_m$, which is $\mathbb{D}$. The proof is complete.

Equicontinuity

Theorem (Arzela-Ascoli)

Let (X, d) *and* (Y, d') *be metric spaces, with* (X, d) *separable, and let* $\mathcal{F}$ *be a family of maps* $f : X \to Y$*.*
Then $\mathcal{F}$ *is normal in* X *if and only if*

(a) *for each* x*,* $\{f(x); f \in \mathcal{F}\}$ *is relatively compact, and*

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With the correct (i.e. non-Euclidean) metrics, normal families are locally uniformly Lipschitz (and not just equicontinuous).

$\mathrm{PSL}(2,\mathbb{C})$ and three dimensional hyperbolic space

The group $\mathrm{PSL}(2,\mathbb{C})$ of Möbius maps

$$z \mapsto (az+b)/(cz+d), \quad ad-bc \neq 0,$$

acts on the complex plane as a subgroup (of index two) of the group generated by inversions in all circles and straight lines.

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It follows that $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ and is, in fact, *the group of conformal isometries of* $\mathbb{H}^3$ *equipped with the hyperbolic metric*

$$ds = \frac{|dx|}{x_3}, \quad x = (x_1, x_2, x_3).$$

Thank you for your attention.