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Key references: books of [Kellogg, Tsuji], Ransford, W. Fuchs. W. K. Hayman .... $(n = 2)$

We have $z = re^{i\theta} = x + iy$.

Let's begin traditionally:

Definition

A real function $u(z)$ is harmonic ($u \in \mathcal{H}$) in the domain $D \subset \mathbb{R}^2$ if u is C^2 and

$$\Delta u := u_{xx} + u_{yy} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0.$$

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A real function $v(z)$ in a domain D is subharmonic if

- v is upper semi-continuous (on each compact set, v is the limit of a decreasing sequence of continuous functions
- If $z_0, B(z_0, h) \in D$ and $v \leq u$ on $S(z_0, r)$, with u harmonic, then

$$v(z) \leq u(z) \qquad (z \in B(z_0, r))$$

An alternative formulation:

Let u be a locally integrable function on a domain $D \subset \mathbb{C}$. Then

- u is harmonic if as a distribution $\Delta u = 0$ on D:

$$\int u \Delta \varphi = 0, \forall \varphi \in \mathcal{D};$$

- u is subharmonic if $\Delta u \geq 0$ [by Riesz: Δu 'is' a (nonnegative measure!)]

$(*)$A locally integrable function u whose distribution Laplacian vanishes as a linear functional coincides a. e. with a (smooth) harmonic function; if u is locally integrable with $\Delta u \geq 0$ as a distribution, then there is a subharmonic function $\tilde{u}$ which agrees with u as a distribution. In my view, this principle motivates many computations one sees frequently.

Key properties:

- Maximum principle **fundamental!**
- Poisson formula (in $\Delta = \{|z| \leq 1\}$):

$$\begin{aligned} u(re^{i\theta}) &= \tfrac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) \tfrac{1-r^2}{1+r^2-2rcos(\theta-\varphi)}\, d\varphi \\ &\qquad (= PI(u, S(1))) \\ &:= \tfrac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) P(1, r, \theta - \phi)\, d\varphi \end{aligned}$$

I used to find this formula ugly, but one can extract gold from it!

- solve the equation $\Delta u = 0, u|\partial\Delta = f$
- $u \in \mathcal{H} \to u \in C^\infty, C^\omega$
- $\mathcal{H}$ closed under *normal convergence.*

Theme for this talk: The maximum (minimum) principle is 'best possible', but a guiding principle for many decades is to try to do more if we have additional information.

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$u(z) = \log|z|$ in $\{0 < |z| < 1\}$. But this is usually treated as a complication: rather, this is the most important harmonic function (punctured disk) or subharmonic function ($\mathbb{C}$).
Rigidity: analytic > harmonic > subharmonic
Remark. The significance of the function $\log|z|$ ($\log|z - z_0|$) is that it is the fundamental solution in $\mathbb{R}^2$ to Laplace's equation:

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This enables us to solve the equation $\Delta u = g$ in D as well (not needed here)–one solution if Ω nice, $\Omega \subset\subset D$:

$$u(z) = (1/2\pi) \int_{\Omega(\zeta)} g(\zeta) \log|z - \zeta|$$

(special case of Riesz decomposition); fundamental solution different when $n > 2$.

RIESZ DECOMPOSITION: *u subharmonic in D and $D' \subset\subset D$. Then there is a* potential *p_μ with support in D' so that on D'*

$$u(z) = (1/2\pi)p_\mu(z) + h(z) \qquad (z \in D')$$

where h is harmonic. (In fact on D' the laplacian of the two sides agree)

$n > 2$: fundamental solution is $c(n)|x|^{-n-2}$, so when $n \neq 2$ is of one sign.

A fundamental connection between complex analysis and potential theory (so $n = 2$ is special in another way) is

(a) f analytic $\rightarrow \log|f|$ is subharmonic;
(a') f meromorphic $\rightarrow \log|f|$ is δ-subharmonic (Δu is a *charge*)

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Converse 'almost' true:

(b) let u be a δ-subharmonic function in D with the property that Δu consists of a discrete set of 'point masses' with weights multiples of $\pm 2\pi$. Then there is a meromorphic function f with $u = \log|f|$.

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Remark Has significance in approximation theory: δ-subharmonic functions are easy to construct. But of doubtful use in complex dynamics.

Consequences

1. Harnack's principle for positive harmonic functions $\mathcal{P}$
Poisson formula: $u(z) = PI(u, S(1))$; if $u|S(1) \geq 0$, since $u(0)$ is the average of u on $S(1)$, then given $r < 1$, Poisson kernet $\rightarrow \exists\, \kappa(r) > 0$ with

$$\kappa(r) \inf_{S(r)} u(\zeta) \leq u(0) \leq (1 - \kappa(r)) \sup_{S(r)} u(\zeta).$$

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Let $\mathcal{P}$ be the cone of positive functions u in the domain D which vanish on ∂D (when D has a nice boundary, technical formulations are ignored here since they do not arise in the next talk)
Remark. If D is bounded, $\mathcal{P} = \varnothing$

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REMARK. If D is bounded, $\mathcal{P} = \varnothing$
APPLICATION FOR RIPPON LECTURE: The study of multiply-connected wandering domains of entire functions—getting large annuli contained in wandering domains.

Simplest examples: $u(z) = y$ in $\{\Im z > 0\}, u(z)$ the Poisson kernel on B.

If we apply the inequality

$$\kappa(r) \inf_{S(r)} u(\zeta) \leq u(0) \leq (1 - \kappa(r)) \sup_{S(r)} u(\zeta).$$

to a family of functions in $\mathcal{P}$ we obtain the
PROPOSITION 1: Let $\{u_n\}$ be a sequence of positive harmonic functions in a domain $D \subset \mathbb{C}$. Then there is a subsequence $\{u_{n_k}\}$ which converges 'normally' on D either to a harmonic function or a constant $0 \leq c \leq \infty$.

Remark. If we have a normalization $u(z_0) = 1$ for some fixed z_0 then $\mathcal{P}$ is compact: the limit function cannot be constant.

Example Martin compactification of a domain $D \subset \mathbb{R}^n$, $n \geq 2$.
Take a fixed point $z_0 \in D$, let $R_n \to \infty$ and D_n be the component of $D \cap B(R_n) \ni z_0$.
For each take boundary values $u(z) = 0$ $(\zeta \in E \cap D_n)$, and $u_n(\zeta) \geq 0$ on $S(R_n)$ to force:

$$u_n(z_0) \equiv 1.$$

Thus as $R_n \to \infty$, and the boundary values on $S(R_n)$ vary, Harnack always gives limit function $u =\in \mathcal{P}$.
But these limits may or may not coincide (thus dim$\mathcal{P}$ an issue).

For example, let D be the plane with some segments $E \subset \mathbb{R}$ deleted. If E is sufficiently dense, $\mathcal{P}$ has two distinct elements, compactified by the ideal point $+i\infty$ or $i - \infty$. If E is not too dense, $\mathcal{P}$ is one-dimensional.

1. the cone $\mathcal{P}$

2. The Dirichlet Problem

Let $D \subset \mathbb{C}$ be a domain, u harmonic in D, $z_0 \in D$.

PROBLEM: Find u with $\Delta u(z) = 0, z \in D, u_{\partial D} = f$.

Poisson formula gives solution in disk; then things get rococo.

Modern 'solution' method of O. Perron via subharmonic functions.

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Modern 'solution' method of O. Perron via subharmonic functions.
MOTIVATION: Case $\mathbb{R}^1$: solve $\Delta u = 0,\ u_{\partial D} = f$ by taking *maximal element* of family of functions with $u'' \geq 0$ [$\sim \Delta u \geq 0$] *whose boundary values are dominated by the desired ones. This is always a 'solution'.*

Example: $D = \{0 < |z| < 1\}$ *boundary values* $u(e^{i\theta}) \equiv 0,\ u(0) = 1$. *The Perron solution in D is* $u(z) \equiv 0$.
subharmonic functions also satisfy the maximum principle: if $z_0 \in D$,

$$u(z_0) \leq \max_{\partial D} u(\zeta),$$

equality only if u is constant.

Remark. One can get a fairly concrete formula for the solution in nice domains via
Green function: $z_0 (\neq \infty) \in D$:

- $g(z, z_0)(= g_D(z, z_0)$ harmonic in $D \setminus z_0$;
- $g(z, z_0) + \log|z - z_0|$ is harmonic at z_0,
- $g = 0$ on ∂D.

If D has a Green function and ∂D is smooth enough, there is the formula

$$u(z_0) = \int_{\partial D} h(\zeta)(\partial g/\partial\eta)(\zeta, z_0)\, ds$$

1. The cone $\mathcal{P}$; 2. The Dirichlet Problem

3. Harmonic measure

D a nice bounded domain, $D \subset \mathbb{C}, z_0 \in D$. For $\varphi \in C(\partial D)$, let h_φ solve the Dirichlet problem in D.

The map

$$h \in C(\partial D) \to u(z_0),$$

where is a *positive linear functional.* F. Riesz: $\exists$ [positive] $\mu = \mu_{z_0}$

with $$u(z_0) = \int_{\partial D} u(\zeta)\, d\mu_{z_0}(\zeta) \; \forall u \in \mathcal{H}.$$

This is the *harmonic measure* at z_0; different z_0 give mutually absolutely continuous measures on ∂D.

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Note: This leads to the notation $\omega(z, E, D)[z \in D; E \subset \partial D]$:

$$\omega(z, E, D) = \int_E d\mu_z(\zeta, D).$$

The value at z of the h. f. u with boundary values

- $u(\zeta) = 1 \ (\zeta \in E)$
- $u(\zeta) = 0 \ (\zeta \in \partial D \setminus E)$.

The ABC-theorem (Eremenko lecture) which bounds the number of direct singularities of a meromorphic function in terms of its (lower) growth depends crucially on an estimate of the harmonic measure of the set

$$I(r, D_j) = \{\theta; re^{i\theta} \in D_j\};$$

Too many $\{D_j\}$ produce j with $I(r_{n(j)}, D_j)$ $(\{n(j)\} \uparrow \infty)$ small for sequence $\{n(j)\} \uparrow \infty$. Then for any $z \in D_j$, $\omega(z, I(r_{n(j)}, D_j \cap B(r_{n(j)}))$ must be extremely small. A form of this is the key to our final slide.

4. Extended Maximum principle.

PROBLEM Let $u \neq const.$ be (sub-)harmonic in D and $u \leq 0$ a. e. on ∂D. Is $u < 0$ in D?

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PROBLEM Let $u \neq const.$ be (sub-)harmonic in D and $u \leq 0$ a. e. on ∂D. Is $u < 0$ in D?

False!: recall

$$u(re^{i\theta}) = PI(u(e^{i\theta}) = \int_0^{2\pi} P(1, r, \theta - \varphi)\, d\varphi$$

Certainly P is harmonic $(P(z) = \Re(1+z)/(1-z))$, and $P(e^{i\theta}) = 0$ for $\theta \neq 0$.

THEOREM 1 (R. NEVANLINNA) *Let u be harmonic in D with $\partial D = E \cup F$, $u(\zeta) \leq M\, (\zeta \in E), u(\zeta) \leq m'(\zeta \in F)$. Then*

$$u(z) \leq M\omega(z, E, D) + N\omega(z, F, D).$$

COROLLARY *If $\omega(z, E, D) = 0$, then we may ignore values on E provided!* u is bounded. What sets have this property?

For this talk, I will not define a general polar set. In physical terms, a polar set is one which is too small to hold an electric charge, and it is defined in terms of E supporting a measure μ whose *energy* is finite. For our purposes we discuss (Ransford, Theorem 3.5.1) the

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THEOREM *Let u be subharmonic in D*, $u \not\equiv -\infty$. *Then* $\{z \in D; u(z) = -\infty\}$ *is a* G_δ *polar set.* (The converse is true also, but not proved in that reference)

Thus polar sets play the role of sets of measure 0 in Lebesgue's theory.

THEOREM (EXTENDED MAXIMUM PRINCIPLE: *Let D a domain whose total boundary is not a polar set [of course* ∂D *is a closed set!], let u be subharmonic on D and bounded above. Suppose there is a polar set* $E \subset \partial D$ *so that*

$$\limsup_{z\to\zeta} u(z) \leq 0 \textit{ for } \zeta \notin E.$$

Then

$$u(z) \leq 0, \forall z \in D.$$

'Subordination Principle' for harmonic measure

This type of maximum principle has been used in an APPLICATION IN NEXT LECTURE: if U is an escaping wandering domain of the entire function f, then 'almost all' points of ∂U escape. (We use domains in $\mathbb{C}$ rather than on the sphere)

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THEOREM: *Let B_1, B_2 be Borel subsets of $\partial D_1, \partial D_2$ where D_1 and D_2 are domains whose boundaries are not polar. Let*

$$f : (D_1, B_1) \to (D_2, B_2) :$$

$[f(D_1) \subset B_2, f(B_1) \subset B_2)]$ *be continuous on the closure, analytic inside. Then*

$$\omega(z, B_1, D_1) \leq \omega(f(z), B_2, D_2)$$

with equality if f is a homeomorphism of the relevant sets.

PROOF: maximum principle: compare $v = \omega(f(z), B_2, D_2)$ and $V = \omega(f(z), f(B_1), f(D_1))$ on D_2.

1. the cone $\mathcal{P}$; 2. Dirichlet Problem; 3. Harmonic measure; 4. Extended Max. Prin

5. Milloux-Schmidt/ Beurling refinement.

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Definitions: u is subharmonic in Δ; for $0 < r < R(= 1)$ let

$$A(r, u) = A(r) = \inf_{S(r)} u(z); \;\; B(r) = \max_{B(r)} u(z).$$

THEOREM (1933): suppose $A(r) < 0 = u(0)$ for all $0 < r < 1$. Then

$$B(r) \leq \frac{4}{\pi} B(1) \tan^{-1}(r/R)^{1/2} \; (0 < r < R).$$

This problem arose [for entire functions]: if $B(r, u) = O(r^{\lambda - o(1)})$ for some $0 \leq \lambda < 1$, then given $\varepsilon > 0$, $\exists \{r_n\} \to \infty$:

$$A(r) > (\cos \pi\lambda - \varepsilon) A(r).$$

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Note. When $\lambda > 1$ the situation is far more complicated; see early papers of Hayman, later work of A. Fryntov

Beurling's form (from his thesis)
Situation: $D = \Delta \setminus \Gamma$, where Γ, Γ consists of a finite number of continua: $D \setminus \Gamma$ multiply-connected. Let E be the *circular projection* of Γ onto (the negative) ray and $D^* = \Delta \setminus E$.
THEOREM *Let* $0 < r_1 < r_2 < 1$ *be given. Then if* $E(r_1, r_2) = E \cap \{r_1 < |z| < r_2\}$ *and*

$$\mu_\ell(E(r_1, r_2)) := \int_{E(r_1,r_2)} t^{-1} dt,$$

then

$$\omega(z, \Gamma, D) \leq \omega(r, E, D^*) < 2e^{-(1/2)\mu_\ell(r,R)}.$$

APPLICATION IN NEXT TALK: Comparison of rates of escape for specific subsets of the Eremenko set $I(f)$ where f is entire