

A directed introduction to harmonic functions

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Key references: books of [Kellogg, Tsuji], Ransford, W. Fuchs. W. K. Hayman ($n = 2$)

We have $z = re^{i\theta} = x + iy$.

Dramatis Personae: $B(r), S(r), D$

Let's begin traditionally:

Definition

A real function $u(z)$ is harmonic ($u \in \mathcal{H}$) in the domain $D \subset \mathbb{R}^2$ if u is C^2 and

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A real function $v(z)$ in a domain D is subharmonic if

- v is upper semi-continuous (on each compact set, v is the limit of a decreasing sequence of continuous functions)
- If $z_0, B(z_0, h) \in D$ and $v \leq u$ on $S(z_0, r)$, with u harmonic, then

$$v(z) \leq u(z) \quad (z \in B(z_0, r))$$

An alternative formulation:

Let u be a locally integrable function on a domain $D \subset \mathbb{C}$. Then

- u is harmonic if as a distribution $\Delta u = 0$ on D :

$$\int u \Delta \varphi = 0, \forall \varphi \in \mathcal{D};$$

- u is subharmonic if $\Delta u \geq 0$ [a (nonnegative measure!)]

(*) A locally integrable function u whose distribution Laplacian vanishes as a linear functional coincides a. e. with a (smooth) harmonic function; if u is locally integrable with $\Delta u \geq 0$ as a distribution, then there is a subharmonic function \tilde{u} which agrees with u as a distribution. In my view, this principle motivates many computations one sees frequently.

Key properties:

- Maximum principle **fundamental!**
- Poisson formula (in $\Delta = \{|z| \leq 1\}$):

$$\begin{aligned}u(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} d\varphi \\ & \quad (= Pl(u, S(1))) \\ & := \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) P(1, r, \theta - \varphi) d\varphi\end{aligned}$$

I used to find this formula ugly, but one can extract gold from it!

- solve the equation $\Delta u = 0, u|_{\partial\Delta} = f$
- $u \in \mathcal{H} \rightarrow u \in C^\infty, C^\omega$
- \mathcal{H} closed under *normal convergence*.

Theme for this talk: The maximum (minimum) principle is 'best possible', but a guiding principle for many decades is to try to do more if we have additional information.

In two dimensions, a harmonic function is 'locally' the real part of an analytic function; caveat:

$u(z) = \log |z|$ in $\{0 < |z| < 1\}$. For us (informally),

Rigidity: analytic $>$ harmonic $>$ subharmonic

Remark. The significance of the function $\log |z|$ ($\log |z - z_0|$) is that it is the fundamental solution in \mathbb{R}^2 to Laplace's equation:

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This enables us to solve the equation $\Delta u = g$ in D as well (not needed here)—one solution if Ω nice, $\Omega \subset\subset D$:

$$u(z) = (1/2\pi) \int_{\Omega(\zeta)} g(\zeta) \log |z - \zeta|$$

(special case of Riesz decomposition); fundamental solution different when $n > 2$.

A fundamental connection between complex analysis and potential theory (so $n = 2$ is special in another way) is

(a) f analytic $\rightarrow \log |f|$ is subharmonic;

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Converse 'almost' true:

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Remark Has significance in approximation theory: δ -subharmonic functions are easy to construct. But of doubtful use in complex dynamics.

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APPLICATION: The study of multiply-connected wandering domains of entire functions—getting large annuli contained in wandering domains.

Simplest example: $u(z) = y$ in $\{\Im z > 0\}$.

Since $P > 0$ in Δ , we have that if $u(z)$ is harmonic, positive, then

$$M(r, u) := \sup_{S(r)} u(re^{i\theta}) > \kappa(1 - r) \inf_{S(r)} u(re^{i\theta});$$

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compactness + diagonal argument shows

PROPOSITION 1: Let $\{u_n\}$ be a sequence of positive harmonic functions in a domain $D \subset \mathbb{C}$. Then there is a subsequence $\{u_{n_k}\}$ which converges 'normally' on D either to a harmonic function or a constant $0 \leq c \leq \infty$.

Remark. If we have a normalization $u(z_0) = 1$ for some fixed z_0 then \mathcal{P} is compact.

Example **Martin compactification** of a domain $D \subset \mathbb{R}^n$, $n \geq 2$.

1. the cone \mathcal{P}

2. The Dirichlet Problem

Let $D \subset \mathbb{C}$ be a domain, u harmonic in D , $z_0 \in D$.

PROBLEM: Find u with $\Delta u(z) = 0, z \in D, u_{\partial D} = f$.

Poisson formula gives solution in disk; then things get rococo.

Modern, general 'solution' method of O. Perron via subharmonic functions.

MOTIVATION: Case \mathbb{R}^1 : solve $\Delta u = 0, u_{\partial D} = f$ by taking maximal element of family of functions with $u'' \geq 0$ [$\sim \Delta u \geq 0$] whose boundary values are dominated by the desired ones. This is always a 'solution'.

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Example: $D = \{0 < |z| < 1\}$ boundary values

$u(e^{i\theta}) \equiv 0, u(0) = 1$.

subharmonic functions also satisfy the maximum principle: if $z_0 \in D$,

$$u(z_0) \leq \max_{\partial D} u(\zeta),$$

equality only if u is constant.

Remark. One can get a fairly concrete formula for the solution in nice domains via

Green function: $z_0 (\neq \infty) \in D$:

- $g(z, z_0) (= g_D(z, z_0))$ harmonic in $D \setminus z_0$;
- $g(z, z_0) + \log |z - z_0|$ is harmonic at z_0 ,
- $g = 0$ on ∂D .

If D has a Green function and ∂D is smooth enough, there is the formula

$$u(z_0) = \int_{\partial D} h(\zeta) (\partial g / \partial \eta)(\zeta, z_0) ds$$

1. The cone \mathcal{P} ; 2. The Dirichlet Problem

3. Harmonic measure

The map

$$u \in \mathcal{H} \rightarrow u(z_0)$$

is a *positive linear functional*. F. Riesz: \exists [positive] $\mu = \mu_{z_0}$ with

$$u(z_0) = \int_{\partial D} u(\zeta) d\mu_{z_0}(\zeta) \quad \forall u \in \mathcal{H}.$$

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Note: This leads to the notation $\omega(z, E, D)[z \in D; E \subset \partial D]$:

$$\omega(z, E, D) = \int_E d\omega(z, \zeta, D).$$

The **value** at z of the h. f. u with boundary values

- $u(\zeta) = 1$ ($\zeta \in E$)
- $u(\zeta) = 0$ ($\zeta \in \partial D \setminus E$).

The ABC-theorem which bounds the number of direct singularities of a meromorphic function in terms of its (lower) growth depends crucially on an estimate of the harmonic measure of the set

$$I(r, D_j) = \{\theta; re^{i\theta} \in D_j\};$$

1. The cone \mathcal{P} ; 2. The Dirichlet Problem 3. Harmonic measure

4. Extended Maximum principle.

PROBLEM Let $u \neq \text{const.}$ be (sub-)harmonic in D and $u \leq 0$ a. e. on ∂D . Is $u < 0$ in D ?

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False!: recall

$$u(re^{i\theta}) = P(u(e^{i\theta})) = \int_0^{2\pi} P(1, r, \theta - \varphi) d\varphi$$

Certainly P is harmonic ($P(z) = \Re(1+z)/(1-z)$), and we see that $P(e^{i\theta}) = 0$ for $\theta \neq 0$.

THEOREM 1 (R. NEVANLINNA) *Let u be harmonic in D with $\partial D = E \cup F$, $u(\zeta) \leq M$ ($\zeta \in E$), $u(\zeta) \leq m'$ ($\zeta \in F$). Then*

$$u(z) \leq M\omega(z, E, D) + N\omega(z, F, D).$$

COROLLARY *If $\omega(z, E, D) = 0$, then we may ignore values on E provided! u is bounded. What sets have this property?*

For this talk, I will not define a general polar set. In physical terms, a polar set is one which is too small to hold an electric charge, and it is defined in terms of E supporting a measure μ whose *energy* is finite. For our purposes we discuss (Ransford, Theorem 3.5.1) the

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THEOREM *Let u be subharmonic in D , $u \not\equiv -\infty$. Then $\{z \in D; u(z) = -\infty\}$ is a G_δ polar set. (The converse is true also, but not proved in that reference)*

Thus polar sets play the role of sets of measure 0 in Lebesgue's theory. When $D = B$, the correspondence is more directly seen.

THEOREM (EXTENDED MAXIMUM PRINCIPLE: *Let D a domain whose total boundary is not a polar set [of course ∂D is a closed set!], let u be subharmonic on D and bounded above. Suppose there is a polar set $E \subset \partial D$ so that*

$$\limsup_{z \rightarrow \zeta} u(z) \leq 0 \text{ for } \zeta \notin E.$$

Then

$$u(z) \leq 0, \forall z \in D.$$

'Subordination Principle' for harmonic measure

This is a type of maximum principle and a special case has been used in an

APPLICATION: if U is an escaping wandering domain of the entire function f , then 'almost all' points of ∂U escape. (We need special case with domains in \mathbb{C} rather than on the sphere)

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THEOREM: *Let B_1, B_2 be Borel subsets of $\partial D_1, \partial D_2$ where D_1 and D_2 are domains whose boundaries are not polar. Let*

$$f : (D_1, B_1) \rightarrow (D_2, B_2)$$

be continuous on the closure, analytic inside. Then

$$\omega(f(z), B_1, D_1) \leq \omega(z, B_2, D_2)$$

with equality if f is a homeomorphism of the relevant sets.

1. the cone \mathcal{P} ;
2. Dirichlet Problem;
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4. Extended Max. Prin
5. Milloux-Schmidt/ Beurling refinement.

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Definitions: u is subharmonic in Δ ; for $0 < r < R(= 1)$ let

$$A(r, u) = A(r) = \inf_{S(r)} u(z); \quad B(r) = \max_{B(r)} u(z).$$

THEOREM (1933): suppose $A(r) < 0 = u(0)$ for all $0 < r < 1$.
Then

$$B(r) \leq \frac{4}{\pi} B(1) \tan^{-1}(r/R)^{1/2} \quad (0 < r < R).$$

This problem arose [for entire functions]: if $B(r, u) = O(r^{\lambda-o(1)})$ for some $0 \leq \lambda < 1$, then given $\varepsilon > 0$, $\exists \{r_n\} \rightarrow \infty$:

$$B(r) > (\cos \pi \lambda - \varepsilon) A(r).$$

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Note. When $\lambda > 1$ the situation is far more complicated; see early papers of Hayman, later work of A. Fryntov

Beurling's form (from his thesis)

Situation: $D = \Delta \setminus \Gamma$, where Γ , Γ consists of a finite number of continua. Let E be the *circular projection* of Γ onto (the negative) ray and $D^* = \Delta \setminus E$.

THEOREM Let $0 < r_1 < r_2 < 1$ be given, and choose

$$\mu < M(r_1).$$

Then if $E(r_1, r_2) = E \cap \{r_1 < |z| < r_2\}$ and $\mu_\ell(E(r_1, r_2)) = \int_{E(r_1, r_2)} t^{-1} dt$, then

$$\omega(z, \Gamma, D) \leq \omega(r, E, D^*) < 2e^{-(1/2)\mu_\ell(r, R)}.$$

APPLICATION: Comparison of rates of escape for specific subsets of the Eremenko set $I(f)$ where f is entire