Dynamics and Deformations of Finite Type Analytic Maps

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Definition

An analytic map of complex 1-manifolds

$$f : W \to X$$

is of *finite type* if :

- X is compact,
- f is open,
- f has no isolated removable singularities,
- $S(f)$ is finite.

Here $S(f)$ is the set of *singular values* :

- By definition, $x \in X$ belongs to the complement of $S(f)$ if and only if there exists an *evenly covered* open neighborhood of x.
- In general, $S(f) = \overline{C(f) \cup A(f)}$, where $C(f)$ is the set of *critical values* and $A(f)$ is the set of *asymptotic values*, hence $S(f) = C(f) \cup A(f)$ for f of finite type.

Examples

- Any nonconstant analytic map between compact Riemann surfaces : in particular, any rational map $\mathbb{P}^1 \to \mathbb{P}^1$.
- Finite type maps $\mathbb{C} \to \widehat{\mathbb{C}}$:
  - entire : exp, sin, cos, $\int^z P(\zeta) \exp(Q(\zeta))\, d\zeta$ for polynomials $P, Q, \ldots$
  - meromorphic : tan, $\wp, \ldots$
  - Finite type maps $\mathbb{C}^* \to \widehat{\mathbb{C}^*}$: $z \mapsto az \exp b(z + \frac{1}{z}), \ldots$
- The elliptic modular functions $j, \lambda : \mathbb{H} \to \mathbb{P}^1$.
- *Skinning maps* on boundaries of hyperbolic 3-manifolds.

New maps from old

- Let $f : W \to X$ and $g : Y \to Z$ be finite type maps with $Y \subseteq X$. Then
$$g \circ f : f^{-1}(Y) \to Z$$
is of finite type. In particular, if $W \subseteq X$ then the iterates f^n are of finite type.
- If $f : W \to X$ is a finite type map, and if Z is a connected component of X which intersects the image of f, then the first visit to Z gives a finite type map $f^{\{Z\}} : W^{\{Z\}} \to Z$.
- Let $f : W \to X$ be a finite type map. If f is *postsingularly finite* then the linearizers for repelling periodic points of f are finite type maps $\Lambda \to X$ for appropriate $\Lambda \subseteq \mathbb{C}$. In particular, for a postsingularly finite entire map $\mathbb{C} \to \widehat{\mathbb{C}}$, the linearizers are finite type entire maps $\mathbb{C} \to \widehat{\mathbb{C}}$.
- Let $f : W \to X$ be a finite type map. The parabolic renormalization construction yields a finite type analytic map $\mathcal{W}_f \to \mathcal{X}_f$.

Islands and tracts

Let $f : W \to X$ be a finite type analytic map, and let $B \subseteq X$ be a Jordan domain whose boundary is disjoint from $S(f)$.

- If B is disjoint from $S(f)$ then the components of $f^{-1}(B)$ are Jordan domains : *(simple) islands* over B.
- If $B \cap S(f)$ consists of a single point then every component D of $f^{-1}(B)$ is simply connected. If D is compactly contained in W then D is a Jordan domain (*island*). Otherwise, D is a *tract*.
- The preimage of the boundary of B is dense in the boundary of any tract over B.

Islands property

Theorem

Let $f : W \to X$ be a finite type analytic map. Assume that X is connected, and let $B \subseteq X$ be a Jordan domain whose closure is disjoint from $S(f)$. Suppose further that $W \subseteq Y$, and let U be a connected open subset of Y which intersects ∂W. Then any connected component of $U \cap W$ contains infinitely many islands over B.

Compare this to the Ahlfors Five Islands Theorem for (not necessarily finite type) meromorphic $\mathbb{C} \to \widehat{\mathbb{C}}$.

Fatou and Julia sets

Let $f : W \to X$ be an analytic map. Assume that $W \subset X$, and consider the sequence of iterates $f^n : W_n \to X$: note that

$$X = W_0 \supseteq W_1 \supseteq W_2 \supseteq \dots$$

- The *Fatou set* $\Omega(f)$ consists of all points $x \in X$ possessing an open neighborhood U such that :
  - Either there exists $N \geq 0$ such that $U \subseteq W_n$ for $n \leq N$ and $U \subseteq X \setminus W_n$ for $n > N$,
  - Or $U \subseteq W_n$ for every n, and moreover the family $\{f^n_{|U} : n \in \mathbb{N}\}$ is normal.
- The *Julia set* $J(f)$ is the complement $X \setminus \Omega(f)$.
- $\Omega(f)$ is open and $J(f)$ is closed.
- $\Omega(f)$ and $J(f)$ are invariant : if $x \in W$ then x belongs to $\Omega(f)$ or $J(f)$ if and only if $f(x)$ does.

Typical and exceptional maps

Let $f : W \to X$ be an analytic map, where X is connected. We say that f is *typical* if $\bigcap_{n=1}^{\infty} W_n$ contains a nonhyperbolic Riemann surface, and *exceptional* otherwise.

Proposition

- *If f is typical then $J(f) = \overline{\bigcup_{n=1}^{\infty} \partial W_n}$.*
- *If f is exceptional then, up to analytic conjugacy, f is :*
  - *(algebraic) a rational endomorphism of $\mathbb{P}^1$, or an affine toral endomorphism, or an automorphism of a higher genus surface ;*
  - *(transcendental) a map $\mathbb{C} \to \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$, or a map $\mathbb{C}^* \to \mathbb{C}^* \hookrightarrow \widehat{\mathbb{C}^*}$, or a map $\mathbb{C} \to \widehat{\mathbb{C}}$ whose second iterate is a map $\mathbb{C}^* \to \mathbb{C}^* \hookrightarrow \widehat{\mathbb{C}^*}$.*

Stratification of $J(f)$

Let $f : W \to X$ be a finite type analytic map.
The islands property has the following consequences :

Proposition

Let $f : W \to X$ be a finite type analytic map.

1. *$J(f)$ is the disjoint union of the sets $f^{-(n-1)}(\partial W)$, for $n \geq 1$, and the set $J_+(f) = J(f) \cap \bigcap_{n=1}^{\infty} W_n$.*
2. *For each component Z of X, either $Z \subseteq J(f)$ or else $J(f)$ is nowhere dense in Z.*
3. *$J(f)$ is an uncountable perfect set, if X is connected and f is not an automorphism.*

Density of repelling periodic points

Theorem

Let $f : W \to X$ be a finite type analytic map. Assume that X is connected. Then ∂W is the accumulation of the set of repelling fixed points of f.

Corollary

Under these assumptions, $J(f)$ is the closure of the set of repelling periodic points of f.

Classification of Fatou components

The domain W may have nonempty exterior $X \setminus \overline{W}$, and Fatou components which eventually map to this exterior are said to *escape*. The remaining Fatou components are classified precisely as in the rational case.

Theorem

Let f be a finite type analytic map.

- *Every component of $\Omega(f)$ which does not escape is eventually periodic.*
- *Every periodic component of $\Omega(f)$ is a superattracting, attracting, or parabolic basin, or a rotation domain.*

In short : there are no wandering domains and no Baker domains.

Fatou-Shishikura inequality

Let $f : W \to X$ be a finite type analytic map. By definition :

- $\gamma(f)$ is the number of cycles of periodic points, with multiplicities

$$\gamma_{\langle x \rangle}(f) = \begin{cases} 0 & \text{if } \langle x \rangle \text{ is repelling or superattracting} \\ 1 & \text{if } \langle x \rangle \text{ is attracting or irrationally indifferent} \\ \nu & \text{if } \langle x \rangle \text{ is parabolic-repelling} \\ \nu + 1 & \text{if } \langle x \rangle \text{ is parabolic-attracting or parabolic-indifferent .} \end{cases}$$

- $\hbar(f)$ is the number of Herman ring cycles.
- $\delta(f)$ is the number of infinite tails of postsingular orbits.

Theorem

Let $f : W \to X$ be a finite type analytic map. Then $\gamma(f) + 2\hbar(f) \leq \delta(f)$, provided that no return to any component of X is an automorphism.

Deformation Theory

Let $f : W \to X$ be a finite type analytic map.

- We associate to f a system of deformation spaces $\mathrm{Def}_A^B(f)$, indexed by appropriate pairs (A, B) of finite subsets of X. These turn out to be finite dimensional complex manifolds.
- Functorial construction, from first principles in Teichmüller theory.
- Tranversality assertions for dynamically natural subloci of these spaces are easy to phrase, study, and prove.
- In the special case of rational maps where one has the concrete parameter spaces Rat_D, these assertions are still easier to first address in the abstract deformation spaces, and then transfer to parameter space by means of a tautological map which is often verifiably nonsingular. The same can be said for those (rare) transcendental maps which belong to evident concrete parameter spaces, namely entire maps $\int^z P(\zeta)\exp(Q(\zeta))\,d\zeta$ for polynomials P, Q, meromorphic maps with rational schwarzian derivative, ...

Teichmüller Spaces

Let X be a compact oriented real 2-manifold, and let $E \subset X$ be finite.

- The Teichmülller space $\text{Teich}(X, E)$ consists of all equivalence classes of complex structures on X, where structures are identified if they are related via pullback by a homeomorphism which is isotopic to the identity relative to E.
- $\text{Teich}(X, E) \cong \prod_{Z \in \pi_0(X)} \text{Teich}(Z, E \cap Z)$
- $\text{Teich}(X, E)$ is a finite dimensional complex manifold. If X is connected of genus g then

$$\dim \text{Teich}(X, E) = \begin{cases} \max(\#E - 3, 0) & \text{if} \quad g = 0 \\ \max(\#E, 1) & \text{if} \quad g = 1 \\ 3g - 3 + \#E & \text{if} \quad g \geq 2 \end{cases}$$

Serre Duality

If X is a complex 1-manifold then $\mathrm{Teich}(X,E)$ has a basepoint $\bullet$. The cotangent and tangent spaces at $\bullet$ have canonical descriptions in terms of sheaf cohomology :

$$\begin{array}{ccc} T^*_\bullet \mathrm{Teich}(X,E) \times T_\bullet \mathrm{Teich}(X,E) & \longrightarrow & \mathbb{C} \\ \Big\downarrow {\scriptstyle \cong \times \cong} & & \Big\uparrow {\scriptstyle \cong} \\ H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E) \times H^1(X, \Theta \otimes \mathcal{O}_{-E}) & \longrightarrow & H^1(X, \Omega) \end{array}$$

- Ω is the sheaf of germs of holomorphic differential forms
- Θ is the sheaf of germs of holomorphic vector fields

The isomorphism $H^1(X,\Omega) \to \mathbb{C}$ is given in terms of a residue sum.

Such a cohomological discussion is available over any algebraically closed field of characteristic zero, for example $\overline{\mathbb{Q}}$.

Dolbeault Isomorphism

$$H^1(X, \Theta \otimes \mathcal{O}_{-E}) \cong \mathrm{Bel}(X)/\mathrm{bel}_E(X)$$

where

$$\begin{aligned} \mathrm{Bel}(X) &= \{(-1,1)\text{-forms on } X\} \\ \mathrm{bel}_E(X) &= \bar{\partial}\{\text{vector fields on } X \text{ which vanish on } E\} \end{aligned}$$

In terms of this description, the pairing

$$H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E) \times H^1(X, \Theta \otimes \mathcal{O}_{-E}) \to \mathbb{C}$$

takes the form

$$(q, [\mu]_E) \mapsto \langle q, \mu \rangle = \frac{1}{2\pi i} \int_X q \cdot \mu$$

Quadratic Differentials

We denote by $\mathcal{Q}(X)$ the $\mathbb{C}$-linear space of all meromorphic quadratic differentials on X with at worst simple poles :

$$\mathcal{Q}(X) = \bigcup_{\text{finite} E \subset X} \mathcal{Q}(X, E)$$

where

$$\mathcal{Q}(X, E) = H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E).$$

- $\mathcal{Q}(X)$ consists of all meromorphic quadratic differentials q on X such that

$$\|q\| = \int_X |q|$$

  is finite.

Forgetful and Pullback Maps

Let A and B be finite subsets of a compact complex 1-manifold X.

- For $A \subseteq B$ there is a *forgetful map*

$$p : \text{Teich}(X, B) \to \text{Teich}(X, A)$$

  with coderivative the inclusion

$$\mathcal{Q}(X, A) \hookrightarrow \mathcal{Q}(X, B)$$

- For finite type f on X, if $f(A) \cup S(f) \subseteq B$ there is a *pullback map*

$$\sigma_f : \text{Teich}(X, B) \to \text{Teich}(X, A)$$

  with coderivative the pushforward operator

$$f_* : \mathcal{Q}(X, A) \to \mathcal{Q}(X, B)$$

  given by

$$f_* q \quad = \sum_{\text{branches } h \text{ of } f^{-1}} h^* q$$

Deformation Spaces

If $A \cup f(A) \cup S(f) \subseteq B$ then p and σ_f share domain and codomain.

$$\mathrm{Def}_A^B(f) \dashrightarrow \mathrm{Teich}(X,B) \rightrightarrows \mathrm{Teich}(X,A)$$

$$\mathrm{Def}_A^B(f) = \{\tau \in \mathrm{Teich}(X,B) \ : \ \sigma_f(\tau) = p(\tau)\}.$$

Theorem

Assume that

$$(\star) \begin{cases} f \text{ is a finite type analytic map on a compact Riemann surface } X, \\ f \text{ is not an automorphism, Lattès example or toral endomorphism,} \\ \#A \geq 3 \text{ if } X \text{ has genus 0, and } \#A \geq 1 \text{ if } X \text{ has genus 1,} \\ A \cup f(A) \cup S(f) \subseteq B. \end{cases}$$

Then $\mathrm{Def}_A^B(f)$ is a $\#(B \setminus A)$-dimensional $\mathbb{C}$-analytic manifold.

Contraction Principle

By the Implicit Function Theorem, the above follows from

Proposition (Thurston, Douady-Hubbard, McMullen, E)

Let X and Y be compact Riemann surfaces, and $f : W \to Y$ a finite type analytic map where $W \subseteq X$. For $q \in \mathcal{Q}(X)$:

- $\|f_* q\| \leq \|q\|$.
- *Equality holds if and only if* $f^* f_* q = (\deg f) \cdot q$, *whence* $\deg f < \infty$.

$$\nabla_f = I - f_*$$

Corollary

Under assumptions $(\star)$ *:*

- $\nabla_f : \mathcal{Q}(X) \to \mathcal{Q}(X)$ *is injective,*
- $T^*_\bullet \mathrm{Def}^B_A(f)$ *is canonically isomorphic to* $\mathcal{Q}(X, B)/\nabla_f \mathcal{Q}(X, A)$.

Variational Formulas

- The variation of postsingular cross-ratios may be expressed in terms of appropriate quadratic differentials with simple poles.
- The variation of multipliers may be expressed in terms of appropriate quadratic differentials with double poles. There are further variational formulas, involving higher order poles, associated to quantities of interest for parabolic cycles.
- The annihilation of Herman rings may be expressed in terms of appropriate quadratic differentials with simple poles, allowing rotationally invariant discontinuities along Herman ring cycles.

Transversality principles

Theorem

Let $f : W \to X$ be a finite type analytic map. Assume that no return to a component of X is an automorphism, an affine toral endomorphism, or a Lattès rational map. Then each of the following conditions determines a smooth hypersurface of the appropriate deformation space $\mathrm{Def}_A^B(f)$:

- *postsingular orbit relation,*
- *cycle of periodic points, with given multiplier in $\overline{\mathbb{D}} \setminus \{0\}$,*
- *cycle of Herman rings, with given Brjuno rotation number.*

Moreover, these hypersurfaces are all mutually transverse. Furthermore, in the case of parabolic points, there is a smooth filtration according to the additional invariants.