The fast escaping set for quasiregular mappings

Joint with W.Bergweiler (Kiel) and D.Drasin (Purdue)

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The role of complex analysis in complex dynamics,
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Overview

- Motivate the study of iterating quasiregular mappings.
- Discuss the escaping set and fast escaping set for quasiregular mappings.
- Describe structures that can arise in the fast escaping set, and give examples.
- Indicate directions for further research.

Motivation 1: complex dynamics

- Complex dynamics deals with the iteration of holomorphic and meromorphic functions in the plane. Throughout, f^n will denote the n'th iterate of a function f.
- For an entire holomorphic function, the plane breaks up into two sets: the Fatou set $F(f)$ where we have stable behaviour of the iterates, and the Julia set $J(f)$ where we have chaotic behaviour.
- The Fatou set is open, the Julia set is closed and the Julia set can be characterized in various ways:
  - (i) $J(f)$ is the closure of the repelling periodic points;
  - (ii) for every neighbourhood U of a point $z \in J(f)$ the forward orbit of U covers everything except possibly one point;
  - (iii) if $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$ denotes the escaping set, then $J(f) = \partial I(f)$. This was proved by Eremenko for transcendental entire functions using Wiman-Valiron theory;
  - (iv) if $A(f)$ denotes the fast escaping set, $J(f) = \partial A(f)$.

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Motivation 2: quasiregular mappings

- Recall that holomorphic functions map small circles to small circles. Roughly speaking, quasiregular mappings defined on subsets of $\mathbb{R}^m$ map small spheres to small ellipsoids, with a uniform bound on the eccentricity allowed.
- The distortion of a quasiregular mapping is encoded by $K(f) \geq 1$, and a mapping is called K-quasiregular if $K(f) \leq K$.
- To have a function theory in higher dimensions, we need to allow distortion because the generalized Liouville's Theorem (Reshetnyak, 1967) states that the only holomorphic mappings in higher dimensions are Möbius mappings or constant.
- There are analogues of Picard's Theorem and Montel's Theorem for quasiregular mappings in $\mathbb{R}^m$, proved by Rickman and Miniowitz respectively.

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Quasiregular dynamics

Our two motivating factors are:

- a well developed and currently research active area of iteration of holomorphic functions in the plane, and
- a natural generalization of holomorphic functions to higher dimensions.

It therefore makes sense to consider iterating quasiregular mappings and investigate the similarities and differences with respect to complex dynamics.

Uniformly quasiregular mappings

- The composition of two quasiregular mappings is again quasiregular, although the distortion will typically increase. If there is an upper bound on the distortion of the iterates, i.e. if $K(f^n) \leq K$ for all $n \geq 1$, then f is called uniformly quasiregular, or uqr.
- These were the first quasiregular mappings to be iterated, by Iwaniec and Martin (1996). They showed that there is a direct analogue of the Fatou and Julia sets for uqr mappings by applying Miniowitz's version of Montel's Theorem.
- Uniformly quasiregular versions of power mappings and Lattés mappings have been constructed, but these mappings are in a sense very special.
- In particular, all the known uniformly quasiregular mappings have finite degree.

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Iterating arbitrary quasiregular mappings

- A quasiregular mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ is said to be of polynomial type if $|f(x)| \to \infty$ as $|x| \to \infty$, and otherwise it is said to be of transcendental type.
- Various quasiregular mappings of transcendental type have been constructed, and none of them are known to be uniformly quasiregular.
- It is therefore natural to ask to what extent an iteration theory exists for these mappings in the same spirit as complex dynamics.
- We will focus on transcendental type mappings in this talk, although the iteration of polynomial type mappings has been studied, e.g. Sun-Yang (2001) in the plane and Fletcher-Nicks (2011) in higher dimensions.

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Theorem (Bergweiler-F-Langley-Meyer, 2009)

Let $f:\mathbb{R}^m\to\mathbb{R}^m$ be K-quasiregular of transcendental type. Then $I(f)$ is non-empty, and contains an unbounded component.

This was proved by extending methods of Dominguez, Rippon and Stallard to the quasiregular setting.

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The fast escaping set

- In the same paper an example was given where $I(f)$ contains a bounded component. This implies that Eremenko's question has a negative answer in the quasiregular setting.
- There are examples where $J(f) \subset \partial I(f)$, and the inclusion is strict. It may be that the escaping set is not quite the right set to consider.
- We will consider a subset of $I(f)$ called the fast escaping set, and defined by

$$A(f) = \{x \in \mathbb{R}^m : \exists L \in \mathbb{N}_0 : f^{n+L}(z) \notin T(f^n(B(0,R)))\},$$

where $R > 0$ is large, $B(0,R)$ is the ball of radius R centred at 0, and $T(E)$ is the topological hull of E, i.e. E together with the bounded components of the complement.
- This was introduced by Bergweiler and Hinkkanen for transcendental entire functions, and has been studied by, for example, Rippon and Stallard.

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Theorem (Bergweiler-Drasin-F, 2013)

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Then $A(f)$ is non-empty and every component is unbounded.

The idea of the proof of this theorem comes from the construction in Bergweiler-Fletcher-Langley-Meyer. It turns out there is an alternative characterization of $A(f)$.

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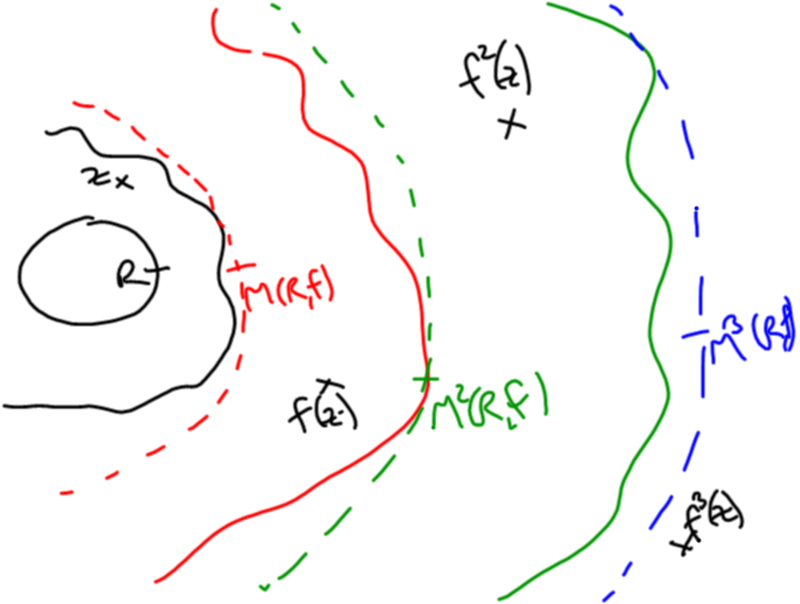

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- Instead, the proof relies on the following weaker annulus covering result.

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Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type and let $\alpha, \beta > 1$. Then, for all large enough r, there exists $R > M(r, f)$ such that

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Structure of $A(f)$: hairs

- There are two structures that are common for $A(f)$.
- The first is hairs: that is, every component of $A(f)$ is a homeomorphic image of a semi-infinite ray.
- It is known (Bergweiler, 2010) that for certain Zorich-type mappings that the escaping set, and hence the fast escaping set, forms hairs. Here, the Zorich map is a higher dimensional version of the exponential function in the plane.
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Structure of $A(f)$: spider's webs

- The other common structure for $A(f)$ is known as a spider's web.
- A set $E \subset \mathbb{R}^m$ is a spider's web if E is connected and there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of bounded topologically convex domains which for each $n \in \mathbb{N}$ satisfy $G_n \subset G_{n+1}$, $\partial G_n \subset E$, and so that $\bigcup_{n\in\mathbb{N}} G_n = \mathbb{R}^m$.

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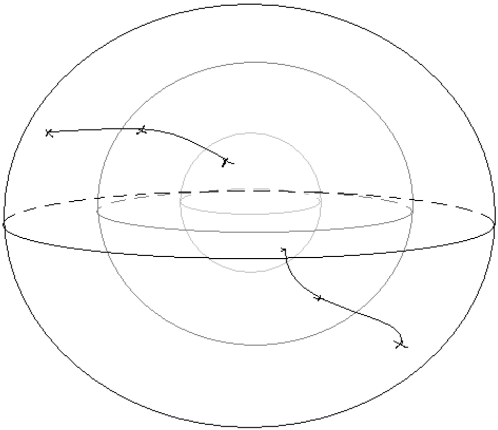

Examples of spider's webs

- There is a class of quasiregular mappings constructed by Drasin and Sastry which satisfies the hypotheses of this theorem. This class gives rise to examples of quasiregular mappings with any of order of growth at least 1.
- The key point about these mappings is that they very often behave like power-type mappings, which means the maximum and minimum modulus is very often comparable.
- Another class of mappings for which $A(f)$ is a spider's web arises from quasiregular linearizers, although we note that these mappings do not satisfy the hypotheses of the previous theorem.
- Mihajlevic-Brandt and Peter showed that if L is a Poincaré linearizer, then $A(L)$ is a spider's web. Here, a Poincaré linearizer locally linearizes a polynomial p at a repelling fixed point z_0 to the map $p'(z_0)z$.

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- There is a class of quasiregular mappings constructed by Drasin and Sastry which satisfies the hypotheses of this theorem. This class gives rise to examples of quasiregular mappings with any of order of growth at least 1.
- The key point about these mappings is that they very often behave like power-type mappings, which means the maximum and minimum modulus is very often comparable.
- Another class of mappings for which $A(f)$ is a spider's web arises from quasiregular linearizers, although we note that these mappings do not satisfy the hypotheses of the previous theorem.
- Mihajlevic-Brandt and Peter showed that if L is a Poincaré linearizer, then $A(L)$ is a spider's web. Here, a Poincaré linearizer locally linearizes a polynomial p at a repelling fixed point z_0 to the map $p'(z_0)z$.

Quasiregular linearizers

- Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a K-uqr mapping with repelling fixed point 0, and suppose that $J(f)$ is a tame Cantor set.
- The infinitesimal space $\mathcal{D}f(0)$ is the set of limit mappings $\lim_{j\to\infty} \lambda_j f(x/\lambda_j)$ where $\lambda_j \to \infty$.
- Since 0 is a repelling fixed point, every element of $\mathcal{D}f(0)$ is a uniformly quasiconformal mapping which is conjugate to $x \mapsto 2x$.
- Using results of Hinkkanen, Martin and Mayer, there exists a quasiregular linearizer $L : \mathbb{R}^m \to \mathbb{R}^m$ of transcendental type which locally conjugates f to $x \mapsto 2x$ near 0.
- Note that there may be many such quasiregular linearizers.

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Further work

(i) Is there a more refined version of the annulus covering theorem used here, in the spirit of Wiman-Valiron theory? Such results use the convexity of $\log M(r, f)$ in $\log r$ for transcendental entire functions. There are examples of quasiregular mappings for which this fails to be true.

(ii) We always have $J(f) \subset \partial I(f)$ for transcendental type maps, but there are examples where the inclusion is strict. On the other hand, it seems plausible that $J(f) = \partial A(f)$.

(iii) Is there a fast escaping set for quasiregular mappings of polynomial type? It is obvious that $A(f) = I(f)$ for polynomials, but polynomial type mappings do not have the same regular growth that polynomials do.

References

- W. Bergweiler, Karpinska's paradox in dimension 3, *Duke Math. J.*, **154** (2010), no. 3, 599-630 (*for examples where $I(f)$ forms hairs*).
- W. Bergweiler, A. Fletcher, J. K. Langley, J. Meyer, The escaping set of a quasiregular mapping, *Proc. Amer. Math. Soc.*, **137** (2009) 641-651 (*for $I(f)$ where f is qr of trans. type*).
- W. Bergweiler, A. Fletcher, D. Drasin, The fast escaping set of a quasiregular mapping, in preparation (*for $A(f)$ where f is qr of trans. type*).
- W. Bergweiler, A. Hinkkanen, On semiconjugation of entire functions, *Math. Proc. Cambridge Philos. Soc.*, **126** (1999), 565–574.
- W. Bergweiler, D. A. Nicks, Foundations for an iteration theory of entire quasiregular maps, to appear in *Israel J. Math.* (*for $J(f)$ where f is qr of trans. type*).
- D. Drasin, S. Sastry, Periodic quasiregular mappings of finite order, *Rev. Mat. Iberoamericana*, **19** (2003), no. 3, 755–766 (*for qr mappings which often behave like power mappings*).

References

- A. Fletcher, Poincaré linearizers in higher dimensions, in preparation.
- A. Fletcher, D. Nicks, Quasiregular dynamics on the n-sphere, *Erg. Th. and Dyn. Sys.*, **31** (2011), 23-31.
- A. Hinkkanen, G. J. Martin, V. Mayer, Local dynamics of uniformly quasiregular mappings, *Math. Scand.*, **95** (2004), no. 1, 80-100.
- T. Iwaniec, G. Martin, Quasiregular semigroups, *Ann. Acad. Sci. Fenn.*, **21** (1996), no. 2, 241–254.
- H.Mihaljevic-Brandt, J. Peter, *Proc. Amer. Math. Soc.*, **140** (2012), 3193-3205.
- P. Rippon, G. Stallard, Fast escaping points of entire functions, *Proc. London Math. Soc.*, **105** (2012), no. 4, 787–820 (*for* $A(f)$ *where* f *is a t.e.f.*).
- D. Sun, L. Yang, Iteration of quasi-rational mapping, *Progr. Natur. Sci. (English Ed.)*, **11** (2001), no.1, 16–25.