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Applications of Nevanlinna theory in complex dynamics

Aimo Hinkkanen

University of Illinois at Urbana–Champaign

aimo@math.uiuc.edu

Topics covered:

Density of repelling periodic points in the Julia set

(Repelling) fixed points of iterates

Permutable entire functions

Factorization of entire and meromorphic functions

Hausdorff dimension of Julia sets (incidental use of the Nevanlinna theory)

Some notation:

If $w \in \mathbb{C}$ and $r > 0$, then

$B(w, r) = \{z \in \mathbb{C} : |z - w| < r\}$,

$\overline{B}(w, r) = \{z \in \mathbb{C} : |z - w| \leq r\}$,

$S(w, r) = \{z \in \mathbb{C} : |z - w| = r\}$.

Iterates of f are denoted by
$f^1 = f$, $f^{n+1} = f \circ f^n$.

We use the usual notation of the Nevanlinna theory.

Density of repelling fixed points in the Julia set

We consider repelling fixed points of the iterates of transcendental entire functions.

If f is entire with iterate f^p, where $p \geq 1$, and if $f^p(z_0) = z_0$ and $|(f^p)'(z_0)| > 1$, then z_0 is called a repelling fixed point of f^p and a repelling periodic point of f.

Theorem. Repelling fixed points of the iterates of a transcendental entire function f are dense in the Julia set $J(f)$.

First proof, due to I.N. Baker (1968) uses a consequence of Ahlfors's Five-Islands Theorem.

Proof by W. Schwick (1997) uses the Nevanlinna theory as well as Zalcman's Lemma (1975).

Zalcman's Lemma. A family $\mathcal{F}$ of meromorphic functions in a domain D is not normal at $w_0 \in D$ if, and only if, there exist a sequence $g_j \in \mathcal{F}$, a sequence $z_j \to w_0$, a positive sequence $\rho_j \to 0$, and a non-constant meromorphic function g on the plane, such that $g_j(z_j + \rho_j z) \to g(z)$ uniformly on compact subsets of $\mathbb{C}$.

Sketch of Schwick's proof.

By Nevanlinna's second fundamental theorem, the set A of points $w \in \mathbb{C}$ such that

$$\Theta(w, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-w}\right)}{T(r, f)} \geq \frac{1}{2}$$

has at most two elements. For any other w, the equation $f(z) = w$ has infinitely many simple roots.

Suppose $w_0 \in J(f) \setminus A$, so $\{f^n : n \geq 1\}$ is not normal at w_0. By Zalcman's Lemma, there are f^{n_j}, $z_j \to w_0$, $\rho_j \to 0$, such that $f^{n_j}(z_j + \rho_j z) \to h(z)$ locally uniformly where h is non-constant entire. Thus $f^{n_j+1}(z_j + \rho_j z) \to (f \circ h)(z)$. Since $w_0 \notin A$, there are distinct a_j, $j \geq 1$, with $f(a_j) = w_0$ and $f'(a_j) \neq 0$.

If h is transcendental then for at least one of a_1, a_2, a_3, say a_1, we have $\Theta(a_1, h) < 1/2$, so there is b with $h(b) = a_1$, $h'(b) \neq 0$.

If h is a polynomial, one of $h(z) = a_1$ or $h(z) = a_2$ has a simple root b, say $h(b) = a_1$.

In both cases, $(f \circ h)(b) = w_0$, $(f \circ h)'(b) \neq 0$. We have

$$f^{n_j+1}(z_j + \rho_j z) - (z_j + \rho_j z) \to (f \circ h)(z) - w_0.$$

The equation $(f \circ h)(z) = w_0$ has the solution b and since the limit function $(f \circ h)(z) - w_0$ is not constant, by Hurwitz's theorem the equation

$$f^{n_j+1}(z_j + \rho_j z) = (z_j + \rho_j z)$$

has a solution c_j for large j with $c_j \to b$. Hence $z_j + \rho_j c_j$ is a fixed point of f^{n_j+1}, with $z_j + \rho_j c_j \to w_0$.

For large j,

$$\begin{aligned} \rho_j (f^{n_j+1})'(z_j + \rho_j c_j) &= (f^{n_j+1}(z_j + \rho_j z))'(c_j) \\ &\to (f \circ h)'(b) \neq 0 \end{aligned}$$

so that $|(f^{n_j+1})'(z_j + \rho_j c_j)| > 1$. Hence $z_j + \rho_j c_j$ is a repelling fixed point of f^{n_j+1}.

Original proof by Baker (1968) is based on a consequence (J. Dufresnoy 1941) of the Ahlfors Five-Islands Theorem (1935):

If D_j, $1 \leq j \leq 5$, are disjoint closed Jordan domains in $\overline{\mathbb{C}}$, there is a constant $C > 0$ depending on the D_j only such that if f is meromorphic in $B(0, R)$ with

$$R|f'(0)|/(1 + |f(0)|^2) > C,$$

then $B(0, R)$ contains at least one domain that f maps conformally onto one of the D_j (if f is analytic, three D_j will do instead of five).

Recall

$$A(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f'(te^{i\theta})|^2}{(1 + |f(te^{i\theta})|)^2} \, t \, dt \, d\theta.$$

We have $T_0(r, f) = \int_0^r A(t, f) \, dt/t$ (the Ahlfors–Shimizu characteristic) and $|T(r, f) - T_0(r, f)|$ is bounded. If f is transcendental, then $A(r, f) \to \infty$ as $r \to \infty$.

Sketch of Baker's proof when f has no Fatou exceptional value:

If $p \in J(f)$ and U is a neighbourhood of p, pick distinct $a_1, a_2, a_3 \in J(f) \cap U$ and disjoint disks $D_j = B(a_j, s) \subset \overline{B}(a_j, s) \subset U$. Determine C from D_1, D_2, D_3. Fix a large R such that $A(R, f) > C^2$. There is $N \in \mathbb{N}$ such that for all $n \geq N$ and $j = 1, 2, 3$, $f^{n-1}(B(a_j, s/3)) \supset \overline{B}(0, R)$, so $f^n(B(a_j, s/3)) \supset f(\overline{B}(0, R))$. Then fix $n = N$. Since area of $B(a_j, s/3)$ is $\pi s^2/9$, there is $b_j \in B(a_j, s/3)$ with

$$|(f^n)'(b_j)|/(1 + |f^n(b_j)|^2) \geq 3C/s.$$

Thus $B(b_j, s/3) \subset D_j$ has a subdomain G_j that f^n maps conformally onto some D_k, $k = k(j)$. This gives a cycle of domains with the end result that for some j, q, f^{qn} maps $G \subset D_j$ conformally onto D_j. Then the proper branch of $(f^{qn})^{-1}$ has an attracting fixed point in G (Rouché's theorem, Schwarz's lemma) and so f^{qn} has a repelling fixed point in $G \subset U$.

Proofs that do not use the Nevanlinna theory but only consequences of Picard's theorem, together with Zalcman's lemma, were given for the density of repelling periodic points in the Julia set for

–a rational or entire function by Berteloot and Duval (2000)

–an analytic function mapping $\mathbb{C}\setminus\{0\}$ into itself by Bargmann (1999)

–a semigroup of rational or entire functions, the semigroup operation being the composition of functions, by Stankewitz (2012)

Repelling fixed points of iterates

We say that z_0 is a periodic point of f of exact (or primitive) order (or period) n if $f^n(z_0) = z_0$ and $f^k(z_0) \neq z_0$ whenever $1 \leq k < n$. If $|(f^n)'(z_0)| > 1$, then z_0 is a repelling periodic point.

W. Bergweiler (1991) answered a question of Baker (1960): If f is a transcendental entire function and $n \geq 2$, then f has infinitely many repelling periodic points of exact period n.

Ex. If $f(z) = z + e^z$ then f has no fixed points.

Bergweiler (1997): If L is a given line then for each $n \geq 2$ there are infinitely many repelling periodic points of f of exact order n that do not lie on L.

Tools used in the study of composite functions

Results for the composition of two functions (often applied so that each function is a suitable iterate of the same function f).

The maximum modulus of f is

$$M(r,f) = \max\{|f(z)| : |z| = r\}.$$

The order of f is

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Pólya's lemma (1926)

If f, g are entire with $g(0) = 0$ and $0 < \rho < 1$, then

$$M(r, f \circ g) \geq M(c(\rho)M(\rho r, g), f)$$

where $c(\rho) = (1-\rho)^2/(4\rho)$ (sharp, Clunie 1970; can be based on Hayman's sharp form (1951) of a result of H. Bohr (1923)).

Sketch of Clunie's proof of Pólya's lemma

Suppose $r = 1$. Let D be the unbounded component of $\overline{\mathbb{C}} \setminus g(S(0,1))$ with boundary curve $\Gamma = \partial D$, $\Delta = \operatorname{int}\Gamma$. May assume $K = 1$ is largest number such that $\overline{B}(0,K) \subset \overline{\Delta}$. Then $M(1,f) \leq \max\{|f(z)| : z \in \Gamma\} \leq M(1, f \circ g)$, so it suffices to show

$$\frac{(1-\rho)^2}{4\rho} M(\rho, g) = c(\rho) M(\rho, g) \leq 1.$$

There is a closed Jordan arc γ from a point of $S(0,1)$ to ∞ that lies in D apart from one end point on $S(0,1)$. Let h map $B(0,1)$ conformally onto $\mathbb{C} \setminus \gamma$ with $h(0) = 0$.

Then g is subordinate to h in $B(0,1)$, so

$$M(\rho, g) \leq M(\rho, h) \leq \frac{|h'(0)|\rho}{(1-\rho)^2}$$

and $|h'(0)| \leq 4$ (Koebe) since h omits a point on $S(0,1)$.

Clunie's estimates for $T(r, f \circ g)$ (1970)

Let f, g be transcendental entire. Then

$$\lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r,f)} = \infty, \qquad \lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r,g)} = \infty,$$

$$\lim_{r\to\infty} \frac{\log M(r, f\circ g)}{\log M(r,f)} = \infty,$$

$$\limsup_{r\to\infty} \frac{\log M(r, f\circ g)}{\log M(r,g)} = \infty.$$

There is an entire g such that

$$\liminf_{r\to\infty} \frac{\log M(r, e^g)}{\log M(r,g)} = 0.$$

So if $1 \le n < m$, then

$$\lim_{r\to\infty} \frac{T(r, f^m)}{T(r, f^n)} = \infty.$$

Bergweiler (1988): f meromorphic, g entire

$$T(r, f\circ g) \le (1 + o(1)) \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f)$$

Results on periodic points

Baker (1959) **(a)** If f is transcendental entire of order $\rho(f) < 1/2$, $n \geq 1$, and $f(z) - z = cz^q + O(z^{q+1})$ as $z \to 0$ where $c \neq 0$, then for a certain constant $k = k(n, \rho(f))$ and all large r,

$$\log\left(\frac{M(r^{1/k}, f^n) - r}{|c|r^q}\right) < N\left(r, \frac{1}{f^n(z) - z}\right).$$

We have, also using Pólya's lemma at last step,

$$\sum_{j=1}^{n-1} N\left(r, \frac{1}{f^j(z) - z}\right) \leq \sum_{j=1}^{n-1} T(r, f^j) + O(\log r)$$
$$= O(T(r, f^{n-1})) = o(\log M(r^{1/k}, f^n)).$$

Thus there any many more fixed points of f^n than of f^j, $j < n$, counting multiplicities. But the fixed points of f^n could still be fixed points of f^j, $j < n$, with a lower multiplicity.

Baker (1959), continued

What can be said about the multiplicities of common fixed points of f^n and some f^j, $j < n$?

We still assume that f is transcendental entire of order $\rho(f) < 1/2$, $n \geq 1$, and $f(z) - z = cz^q + O(z^{q+1})$ as $z \to 0$ where $c \neq 0$.

(b) If, in addition, p is a prime and at no fixed point z_0 of f, we have $f'(z_0) = 1$ or $f'(z_0)^p = 1$, then f has infinitely many periodic points of exact order p.

Baker (1960) If f is transcendental entire, then f has periodic points of exact order n for each $n \geq 1$ except possibly for at most one value of n.

If there are none for n that is minimal, choose $m > n$, set $\varphi(z) = (f^m(z) - z)/(f^{m-n}(z) - z)$. Then

$$T(r, \varphi) \leq T(r, f^m) + T(r, f^{m-n}) + O(\log r)$$
$$= (1 + o(1))T(r, f^m)$$

and from $f^m(z) - z = \varphi(z)(f^{m-n}(z) - z)$, we get

$T(r, f^m) \leq (1 + o(1))T(r, \varphi)$, so

$T(r, \varphi) = (1 + o(1))T(r, f^m)$.

We have

$\overline{N}(r, 0, \varphi) \leq \overline{N}(r, 0, f^m(z) - z)$,

$\overline{N}(r, \varphi) \leq \overline{N}(r, 0, f^{m-n} - z) < T(r, f^{m-n}) + O(\log r)$.

If $\varphi(z) = 1$, then $f^m(z) = f^{m-n}(z)$, so $\xi = f^{m-n}(z)$ satisfies $f^n(\xi) = \xi$, hence $f^j(\xi) = \xi$ for some j with $1 \le j < n$. Thus

$$\overline{N}(r,1,\varphi) \le \sum_{j=1}^{n-1} \overline{N}(r,0,f^{m-n+j} - f^{m-n})$$
$$\le \sum_{j=1}^{n-1} T(r, f^{m-n+j} - f^{m-n})$$
$$\le \sum_{j=1}^{n-1} T(r, f^{m-n+j}) + (n-1)T(r, f^{m-n}) + O(1).$$

By Nevanlinna's second fundamental theorem

$$T(r,\varphi) \le \overline{N}(r,0,\varphi) + \overline{N}(r,1,\varphi) + \overline{N}(r,\varphi) + S(r)$$

where $S(r) = O(\log(rT(r,\varphi)))$ outside a set of finite length. Hence

$$\begin{aligned} T(r,\varphi) &\le \overline{N}(r,0,f^m(z) - z) + nT(r,f^{m-n}) \\ &+ \sum_{j=1}^{n-1} T(r, f^{m-n+j}) + S(r). \end{aligned}$$

Dividing by $T(r, f^m)$ we get

$$1 \leq \liminf_{r\to\infty} \frac{\overline{N}(r, 0, f^m(z) - z)}{T(r, f^m)}.$$

Thus f^m has many distinct fixed points. For fixed points of f^j, $1 \leq j < m$, we have

$$\sum_{j=1}^{m-1} N\left(r, \frac{1}{f^j(z) - z}\right) \leq \sum_{j=1}^{m-1} T(r, f^j) + O(\log r)$$
$$= o(T(r, f^m)).$$

Thus f has infinitely many periodic points of exact order m.

Bergweiler (1991): If f is transcendental entire and $n \geq 2$, then f has infinitely many repelling periodic points of exact order n.

Further, if g, h are transcendental entire then $h \circ g$ has infinitely many repelling fixed points (conjectured by F. Gross 1966 without "repelling").

Major tools used in the proof:

– Wiman–Valiron theory of power series

– a form of Ahlfors Five-Islands Theorem

– polynomial-like mappings

Bergweiler (1997) on periodic points outside a line uses some Nevanlinna theory.

Permutable entire functions

Entire functions f, g are said to be permutable (or commuting) if $f \circ g = g \circ f$.

Open questions:

1. If f, g are permutable, do we have $J(f) = J(g)$? True if f, g are permutable rational functions.

2. If f, g are permutable and distinct and both are of positive finite order, what can be said about them? For example, do we have $R_1 \circ f = R_2 \circ g$ for rational functions R_1, R_2 ?

The lower order of f is

$$\lambda(f) = \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

Some results on permutable transcendental entire functions f, g when f, g are not specified.

Baker (1984): If $g = f + c$ for a constant c, then $J(f) = J(g)$.

Poon and Yang (1998): If $g = af + b$ for constants a, b, then $J(f) = J(g)$.

Wang and Yang (2003): If $q \circ g = a(q \circ f) + b$ for a non-constant polynomial q and constants a, b, then $J(f) = J(g)$.

Ren and Li (1997): If $f, g \in \mathcal{S}$ (finitely many singular (i.e., critical and finite asymptotic) values) then $J(f) = J(g)$.

Hua and Wang (2002): If $f, g \in \mathcal{B}$ (the set of singular values is bounded) then $J(f) = J(g)$. The proof uses Eremenko's escaping set $I(f) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}$.

Some results on specified permutable transcendental entire functions f, g.

X. Hua, R. Vaillancourt and X. Wang (2007): If $\rho(g) < \infty$ and f satisfies

$$\sum_{j=0}^{n} p_j(z) f^{(j)}(z) + p(z) = 0$$

where p and the p_j are polynomials, $p_n, p_0 \not\equiv 0$, p/p_0 is non-constant, then there are non-constant rational functions R_1, R_2 such that $R_1 \circ f = R_2 \circ g$. (It is also claimed that $J(f) = J(g)$.)

For example, f may be of the form

$$f(z) = q(z) + \sum_{j=1}^{n} q_j(z) \exp(r_j(z))$$

for polynomials q, q_j, r_j.

Pick $a \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{N}$. Below, $f \circ g = g \circ f$.

Example. (Hua-Wang-Yang 2000) Set

$$f(z) = ia\left[\exp\left(ibz^2\right) + \exp\left(-ibz^2\right)\right]$$

$$g(z) = a\left[\exp\left(ibz^2\right) - \exp\left(-ibz^2\right)\right]$$

$$q(z) = ibz^2, \qquad b = \frac{(4k+3)\pi}{8a^2}.$$

Then $q \circ g = -q \circ f - \left(2k + \frac{3}{2}\right)\pi i$.

Example. $f(z) = z + ae^z$, $g(z) = z + ae^z + 2k\pi i$.

Example. $f(z) = z + a\sin z$, and
$g(z) = z + a\sin z + 2k\pi$ or
$g(z) = -z - a\sin z + 2k\pi$.

Earlier results due to (e.g.) Poon and Yang (1998).

T.-W. Ng (2001): Let f, g be transcendental entire with $f \circ g = g \circ f$. Then $g(z) = af^n(z) + b$, where a is a root of unity, $n \in \mathbb{N}$, and $b \in \mathbb{C}$, and furthermore $J(f) = J(g)$, provided that all of the following hold.

(i) f is not of the form $H \circ Q$, where H is periodic and Q is a polynomial.

(ii) f is left-prime (if $f = \alpha \circ \beta$ with entire α and β, where β is transcendental, then α is linear).

(iii) f' has at least two distinct zeros.

(iv) There exists $N \in \mathbb{N}$ such that for any $c \in \mathbb{C}$, the equations $f(z) = c$ and $f'(z) = 0$ have at most N common solutions.

(v) The orders of the zeros of f' are uniformly bounded.

For example, we may have $f = p + e^z$ or $f = p + \sin z$ for a non-constant polynomial p. Further results: Liao-Yang 2005.

Techniques of proof in Hua, Vaillancourt and Wang (2007) and many other earlier related papers go back to Jian-Hua Zheng and Zheng-Zhong Zhou (1988). Those techniques make substantial use of the Nevanlinna theory.

Let f, g be permutable transcendental entire of finite order.

If

$$\sum_{j=0}^{n} p_j(z) f^{(j)}(z) + p(z) = 0$$

where p and the p_j are polynomials, not all zero, then for some polynomials q and q_j, not all zero, we have

$$\sum_{j=0}^{n} q_j(z) g^{(j)}(z) + q(z) = 0.$$

The proof is based on a result of N. Steinmetz (1980).

A result of N. Steinmetz (1980)

Let $F_j \not\equiv 0$ and $h_j \not\equiv 0$ be meromorphic in $\mathbb{C}$, $0 \le j \le m$, g non-constant entire, $K > 0$ constant,

$$\sum_{j=0}^{m} T(r, h_j) \le KT(r, f) + S(r, f), \qquad (1)$$

$$\sum_{j=0}^{m} F_j(f(z))h_j(z) \equiv 0.$$

Then there are polynomials P_j, Q_j, $0 \le j \le m$, in each case not all zero, such that

$$\sum_{j=0}^{m} P_j(f(z))h_j(z) \equiv 0,$$

$$\sum_{j=0}^{m} F_j(z)Q_j(z) \equiv 0.$$

The proof uses Nevanlinna's second fundamental theorem and shows that when f is of finite order, it suffices to have (1) in a sequence $r_k \to \infty$.

An application of Pólya's Lemma (Zheng-Zhou 1988).

Since $f \circ g = g \circ f$, $\rho(g) < \infty$ and $\lambda(f) > 0$, we have with $c = c(1/2) = 1/8$ and some $K_1, K_2, K_3 > 0$,

$$\begin{aligned} &M(M(r,f),g) \geq M(r, g \circ f) \\ &= M(r, f \circ g) \geq M(cM(r/2,g),f), \end{aligned}$$

$$\begin{aligned} &K_1 \log M(r,f) > \log\log M(cM(r/2,g),f) \\ &> K_2 \log M(r/2,g) \geq K_2 T(r/2,g). \end{aligned}$$

Since $\log M(r,f) \leq 3T(2r,f)$, we have

$$T(r,g) < K_3 T(4r,f).$$

If $n \geq 1$, by the lemma on the logarithmic derivative and Pólya peaks, get $r_j \to \infty$ with

$$T(r_j, g^{(k)}) \leq K T(r_j, f)$$

for all j, for $0 \leq k \leq n$, and for some $K > 0$.

Differentiating $f \circ g = g \circ f$ several times, substituting $g(z)$ for z in

$$\sum_{j=0}^{n} p_j(z) f^{(j)}(z) + p(z) = 0$$

and performing several eliminations, we obtain

$$h_n \cdot g^{(n)}(f(z)) + \cdots + h_0 \cdot g(f(z)) + h_{n+1}(z) = 0$$

where the h_j are polynomials in $z, f, f', \ldots, f^{(n)}, g, g', \ldots, g^{(n)}$. We have

$$T(r_k, h_j) < KT(r_k, f)$$

for $0 \leq j \leq n+1$ and for some $r_k \to \infty$.

By the result of Steinmetz, there are polynomials Q_j, $0 \leq j \leq n+1$, not all zero, such that

$$\sum_{j=0}^{n} Q_j(z) g^{(j)}(z) + Q_{n+1}(z) \equiv 0,$$

as claimed.

In the proof by Hua, Vaillancourt and Wang (2007), from

$$\sum_{j=0}^{n} p_j(z) f^{(j)}(z) + p(z) = 0, \qquad (2)$$

$$\sum_{j=0}^{n} q_j(z) g^{(j)}(z) + q(z) = 0, \qquad (3)$$

we obtain by substituting $g(z)$ for z in (2) and $f(z)$ for z in (3), and further eliminations that there are non-constant rational functions R_1, R_2 such that $R_1 \circ f = R_2 \circ g$.

Bergweiler (2008): Let f, g be transcendental entire with $f \circ g = g \circ f$. If f satisfies an algebraic differential equation, then so does g.

Factorization of functions

We consider the factorization of entire and meromorphic functions.

Let F, f, g be non-constant entire functions, with F transcendental. If in every representation

$$F = f \circ g$$

at least one of f, g is a polynomial of degree 1, then F is called **prime**. If one of f, g must be a polynomial, then F is called **pseudo-prime**. If f must be linear then F is left-prime, and if g must be linear then F is right-prime, all "in the entire sense".

Not a factorization in the entire sense:
$e^{-2z} = z^{-2} \circ e^z$

Many proofs in factorization theory make significant use of the Nevanlinna theory, for example:

R. Goldstein (1970): Let f be a transcendental entire function of finite order. Suppose that for some finite a we have $\delta(a, f) = 1$. Then f is pseudo-prime.

N. Steinmetz (1980): Any solution f, non-constant and meromorphic in the plane, of

$$f^{(n)}(z) + \sum_{j=0}^{n-1} A_{j+1}(z) f^{(j)}(z) + A_0(z) = 0$$

with rational A_j, is pseudo-prime.

N. Steinmetz (1982): Any solution f of the first or second Painlevé equation is prime:

$$f''(z) = z + 6f(z)^2$$

$$f''(z) = 2f(z)^3 + zf(z) + \alpha,$$

$\alpha \in \mathbb{C}$.

Hausdorff dimension of Julia sets

We consider the Hausdorff dimension of the Julia sets of certain entire functions.

Just one example.

K. Barański (2008):

Let f be a transcendental entire function of finite order with an attracting fixed point z_0 such that the singular set Sing (f) is contained in a compact subset of the immediate basin of attraction $B(z_0)$ of z_0 (e.g., $f(z) = \lambda e^z$, $\lambda \in (0, 1/e)$, or $f(z) = \lambda \sin z$, $\lambda \in (0, 1)$, or $f(z) = \lambda g(z)$, $g \in \mathcal{B}$ (Sing (g) is bounded), $|\lambda|$ small).

Then $J(f)$ consists of hairs, the Hausdorff dimension of the set of end points of hairs is 2, and the Hausdorff dimension of the union of hairs without end points is 1.

At one point in Barański's proof, a certain $\zeta \in \mathbb{C}$ is chosen, and it is noted that (since f is of finite order) for all large t and some $\lambda > 0$ we have

$$T(t, f) < t^{\lambda}$$

and hence

$$N\left(t, \frac{1}{f - \zeta}\right) < 2t^{\lambda}$$

and thus

$$n\left(t, \frac{1}{f - \zeta}\right) < ct^{\lambda}$$

for some $c > 0$. This is then used to estimate the cardinalities of certain sets.

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