Bishop's qc-folding and wandering domains in Eremenko-Lyubich class

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23 May, 2013

Background

Definition: Let f be a rational or entire transcendental map. Let U be a Fatou domain of f. If $f^\ell(U)$, $\ell \geq 0$ is never eventually periodic then we say that U is wandering. In this case we have

$$f^n(U) \cap f^m(U) = \emptyset \quad \forall n < m \in \mathbb{Z}.$$

Theorem (Sullivan 1985): Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let U be a Fatou domain of R. Then $f^\ell(U)$ is eventually periodic for some $\ell \geq 0$. In other words, U cannot be a wandering domain.

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Remark: In what follows f will be an entire transcendental map.

Multiply-connected wandering domain (Baker 1976)

Let $g(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$ where

- $1 < a_1 < a_2 < \dots,$
- $a_{j+1} < g(a_j) < 2a_{n+1},$
- $g(A_j) \subset A_{j+1}$ where

$$A_j = \{z \in \mathbb{C} \mid a_j^2 < |z| < \sqrt{a_{j+1}}\}.$$

Remark: If $A_j \subset U_j$ then $g(U_j) = U_{j+1}$, hence $g^k(U_j) \to \infty$ as $k \to \infty$ for all j.

Theorem (Baker 1975, 1985): If U is a multiply-connected component of the Fatou set of f then U wanders.

Wandering domain example II (Herman-Sullivan's 80's)

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{z-1+e^{-z}+2\pi i} & \mathbb{C} \\
{\scriptstyle e^{-z}}\big\downarrow & & \big\downarrow{\scriptstyle e^{-z}} \\
\mathbb{C}\setminus\{0\} & \xrightarrow{ewe^{-w}} & \mathbb{C}\setminus\{0\}
\end{array} \tag{1}
$$

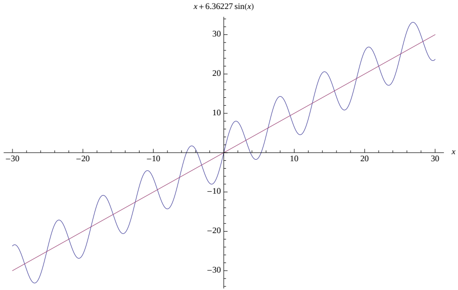
Lemma: If f and g are entire, f and g commutes, and $f = g + c$ for some constant c, then $J(f) = J(g)$.

Application: $f(z) = z - 1 + e^{-z}$ and $g(z) = f(z) + 2\pi i$. So, the Fatou set of g has a wandering domain.

Wandering domain example III

Let $g(z) = z + \lambda_0 \sin(z)$ with $\lambda_0 \approx 6.36227$.

- The (infinitely many) critical points: $\overline{x} = cos^{-1}(-1/\lambda_0)$.
- λ_0 is such that $g(\overline{x}) = \overline{x} + 2\pi i$ for all c.p. $\overline{x}$.
- Only two critical orbits, both belonging to the Fatou set.
- The lines $\{x = k\pi\} \subset \mathcal{J}(g)$ for all $k \in \mathbb{Z}$.



Critically finite entire transcendental functions

We denote by $\mathcal{S}(f)$ the set of (finite) singularities of f^{-1} (critical values, asymptotic values and limits of those values).

Definition: We say that f is critically finite if $\mathcal{S}(f)$ is finite.

$$E_\lambda(z) = \lambda \exp(z) \quad \mathrm{Sn}_\lambda(z) = \lambda \sin(z)$$

Theorem (Baker 1984): If $f(z) = \int_0^z P(t)e^{Q(t)}dt$, P and Q polynomials, then f has no wandering domains. Indeed this follows from a more general statement.

Theorem (Eremenko-Lyubich, Golberg-Keen 1986): If f is critically finite then f has no wandering domains.

Remark: The proofs adapted Sullivan's quasi-conformal strategy.

Constant limit functions

Theorem (Fatou 1920): Let U a wandering domain of f. All limit functions of $\{f^{n_k}|_U\}$ are constant (non constant limit functions correspond to eventually periodic Fatou components).

- $\{f^n|_U\} \to \infty$ (uniformly tending to infinity)
- $\{f^{n_k}|_U\} \to \infty$ and $\{f^{m_k}|_U\} \to a \in \mathcal{J}(f) \subset \mathbb{C}$ (oscillating)
- There is no $\{n_k\}$ for which $\{f^{n_k}|_U\} \to \infty$ (bounded)

Remark: All previous examples were of the first type.

Example (Eremenko-Lyubich 1987): There exits a transcendental entire function f which has an oscillating wandering component U (with infinitely many finite constant limit points).

Remark: As far as I know there are no examples of the third type.

Composite entire functions and Carleman sets

Theorem (Bergweiler-Wang 1998): Let f, g be entire maps. Then

$$z \in \mathcal{J}(f \circ g) \iff g(z) \in \mathcal{J}(g \circ f).$$

If $U_0 \subset \mathcal{F}(f \circ g)$ and $V_0 \subset \mathcal{F}(g \circ f)$ with $g(U_0) \subset V_0$ then

$$U_0 \text{ wanders} \iff V_0 \text{ wanders}$$

Main tool: g semi-conjugates $f \circ g$ with $g \circ f$.

Theorem (Singh 2003): There exists $U \in \mathbb{C}$ and f, g entire maps such that U is periodic for f, g and $g \circ f$ but it wanders for $f \circ g$.

Remark: Carleman approximation theory.

Problem: It is possible, in general, to relate the existence of wandering domains for f, g and $f \circ g$?

Constant limit functions

Definition: We define

$$E = \bigcup_{s \in \mathcal{S}(f)} \bigcup_{n \geq 0} f^n(s)$$

We denote by E' the derived set of E, that is, the set of finite limit points of E. And we denote by $\overline{E}$ the closure of E.

Theorem (Baker, 1976): Let U a Fatou component of f. Then all constant limit function of $\{f^{n_k}|_U\}$ belong to $\overline{E} \cup \infty$.

Theorem (BHKMT, 1993): Let U a Fatou component of f. Then all constant limit function of $\{f^{n_k}|_U\}$ belong to $E' \cup \infty$.

Corollary: The maps (in class $\mathcal{B}$) $f(z) = e^z$, $f(z) = \dfrac{\sin(z)}{z}$, $f(z) = \frac{\pi^2}{\pi^2 - z^2} \sin(z)$ has no wandering domains.

Eremenko-Lyubich class and wandering domains

Definition: We say that f is in Eremenko-Lyubich class, $f \in \mathcal{B}$, if $\mathcal{S}(f)$ is bounded.

Theorem (Eremenko-Lyubich 1985): If $f \in \mathcal{B}$ there are no Fatou components such that $\{f^n|_U\}$ converges uniformly to infinity. No Baker domains and all wandering domains would be, if exist, either oscillating or bounded. Main result $\mathcal{J}(f) = \overline{\mathcal{I}(f)}$.

Theorem (Mihaljević-Rempe 2012): If $f \in \mathcal{B}$, all singular values escape uniformly to infinity and f satisfies *condition A*, then f has no wandering domains.

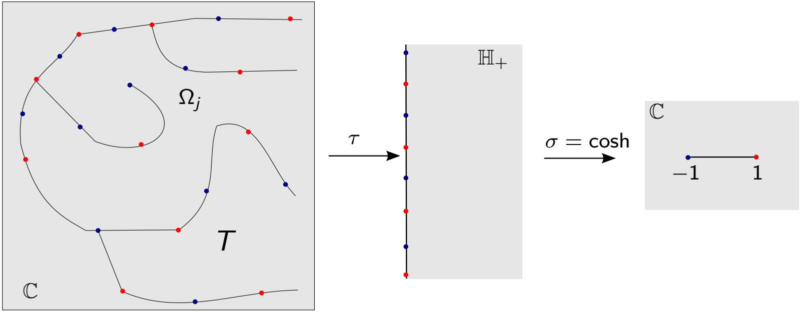
Question: Is it possible to erase *condition A* from the statement? Bishop's example does not answer to this question since not all singular values escape to infinity uniformly.

Bishop's QC-folding for entire maps (in class $\mathcal{B}$)

The following slices explain Christopher Bishop's construction of a wandering domain in class $\mathcal{B}$.

- This corresponds to Section 17 on the paper *Constructing entire functions by quasiconformal folding*.
- I want to thank him for patiently answer my questions while I was reading the paper.
- Of course, possible mistakes or inaccuracies on this exposition, if any, belongs to me.

Assume f is entire with precisely two critical values: $\{-1, 1\}$.
Let $T = f^{-1}([-1, 1])$.



Then we may write $f(z) = \cosh(\tau(z))$

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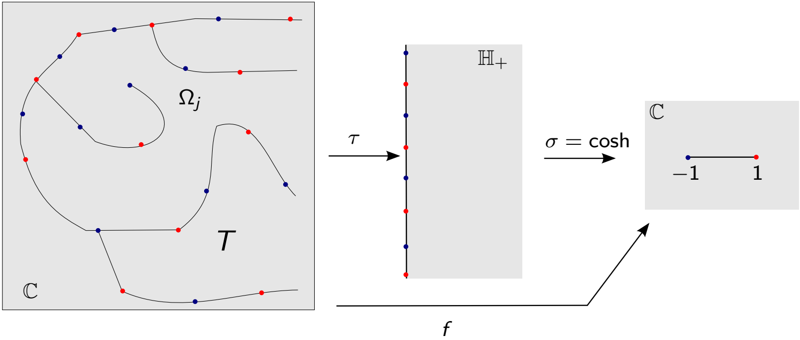

Main problem: Is it possible to start with an abstract tree T and a map τ so that $f = \sigma\tau$ is entire? Answer: Formally no, but close to this.

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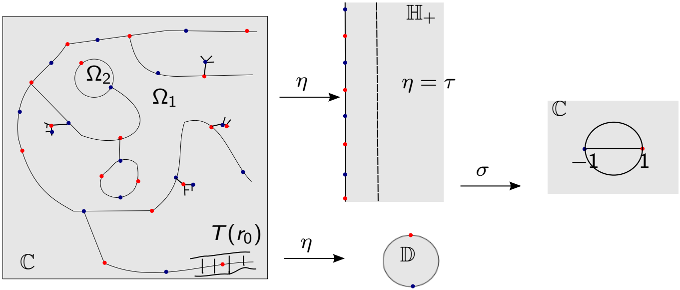


Restriccions:

- T should be of uniformly - M - bounded geometry (smooth + relative size of edges + angle condition).
- No common edges for bounded components. Moreover τ sends vertices of bounded components to roots of unity.
- τ-sizes are $\geq \pi$.

The prize: $\exists\ f = g \circ \phi$, $g = \sigma \circ \eta$ q.r and ϕ K-qc with $K = K(M)$. Moreover $f = \sigma \circ \tau \circ \phi$ off $T(r_0)$.

Bishop's QC-folding for entire maps in class $\mathcal{B}$

Theorem: There exist an entire function $f \in \mathcal{B}$, a disc D_0 and an increasing sequence of integers $\{n_k\} \nearrow \infty$ so that if we set $D_{n+1} = f(D_n), \ n \geq 1$, then

(a) The diameter of D_n tends to zero (not monotonically),

(b) $\mathrm{dist}(0, D_{n_k}) \nearrow \infty$, and

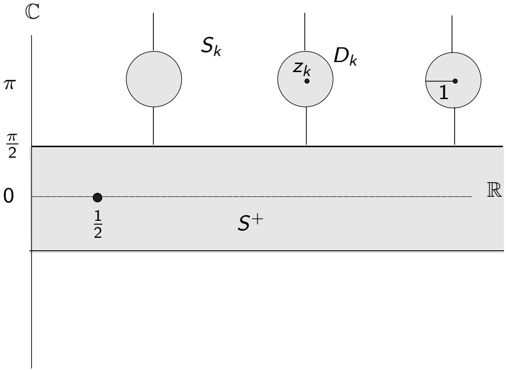
(c) $\mathrm{dist}(0, D_{n_k+1}) < 1$.

In particular, f has a wandering domain.

Proof:

- D_0 and all its images belong to the Fatou set. Set U_0 the Fatou component where D_0 lies.
- U_0 cannot be (pre)periodic because of (b).

Bishop's QC-folding for entire maps in class $\mathcal{B}$



Remark: Symmetry (real and imaginary axis) of the construction.

$$f(\overline{z}) = \overline{f(z)} \quad \text{and} \quad f(-z) = -f(z)$$

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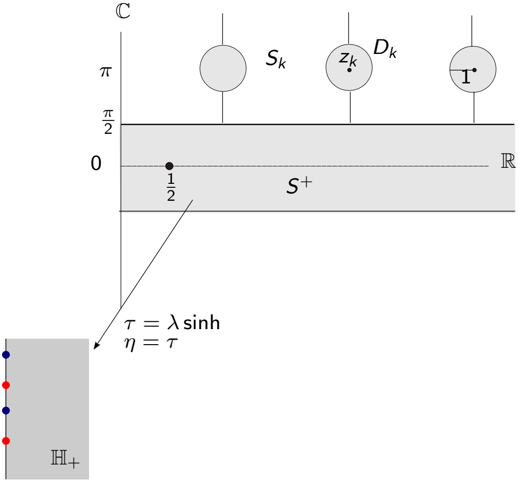

Remark: The blue and red points correspond to $\pi i \mathbb{Z}$.

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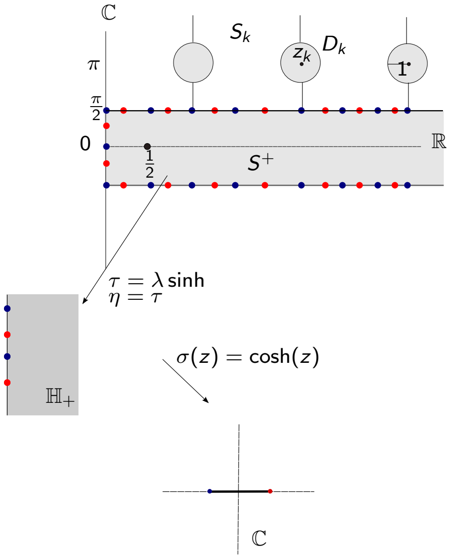

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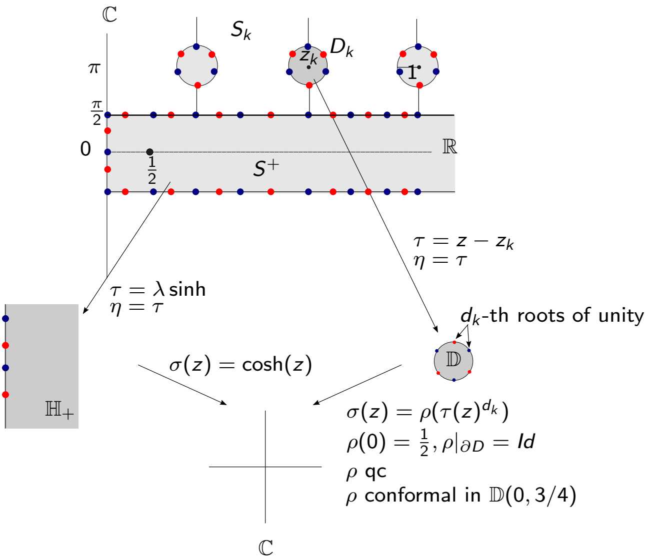

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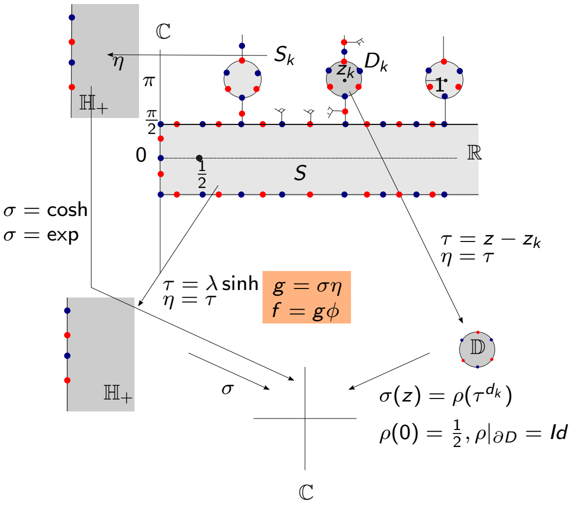

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Important remarks about ϕ:

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- Moreover ϕ is symmetric (1-to-1 on $\mathbb{R}$), $\phi(0) = 0$, $\phi(\infty) = \infty$ and

$$\phi(z) = z + \frac{a}{z} + O\left(|z^{-2}|\right)$$

for some $|z| > R$ (Dyn'kin's Theorem).

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- ϕ' should be bounded by below from 0.
- Estimates get better when increasing the parameters.

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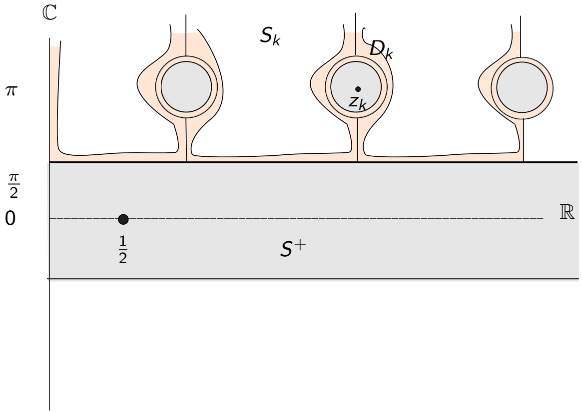

Consequently: $f'(x) = \frac{d}{dx} \cosh\left(\lambda \sinh(\phi(x))\right) \geq 16x$ for λ large enough. Hence we may choose λ so that $x_n = f^n\left(\frac{1}{2}\right)$ tends to infinity (exponential speed) as n does.

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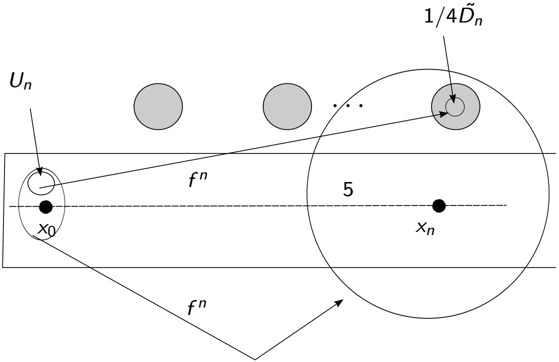
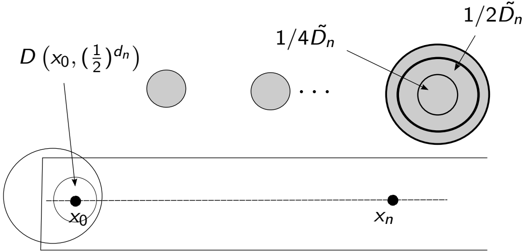


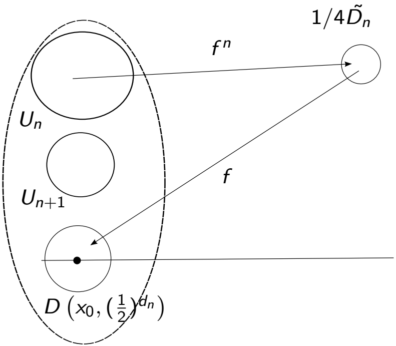
The nth preimage. The disc U_n (1/4 Koebe's Theorem) has radius comparable to

$$r_n = \left(\frac{d}{dx} f^n\left(\frac{1}{2}\right)\right)^{-1}.$$



The image by $f = g \circ \phi$ of $1/4\tilde{D}_n$.

- The map ϕ sends $1/4\tilde{D}_n$ inside $1/2\tilde{D}_n$ (Dyn'kin's estimates for n large enough).
- The map $g(z) = \rho\left((z - \tilde{z}_n)^{d_n}\right)$ sends $D(x_0, (\frac{1}{2})^{d_n})$.



We want $\left(\frac{1}{2}\right)^{d_n} << r_{n+1}$. This is possible since

- ϕ_ℓ is uniformly K-qc for all choices of $\{d_n\}$ and supported in smaller domains as $\{d_n\} \nearrow \infty$.
- There exists a convergence subsequence to a map ϕ_∞ (on compact subsets of $\mathbb{C}$).
- The corresponding f_ℓ also converge on compact subsets of S_+ to f_∞. Hence the r_{n_ℓ} also converge to non-zero limits r_{n_∞}.

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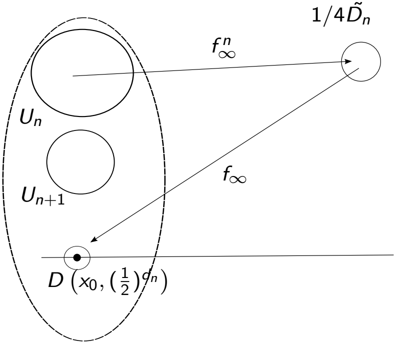

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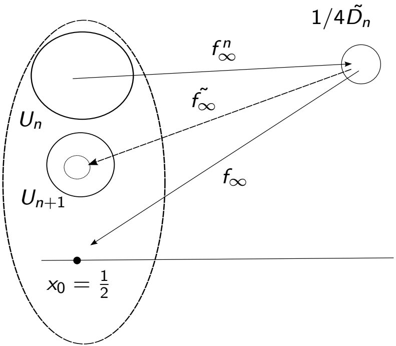

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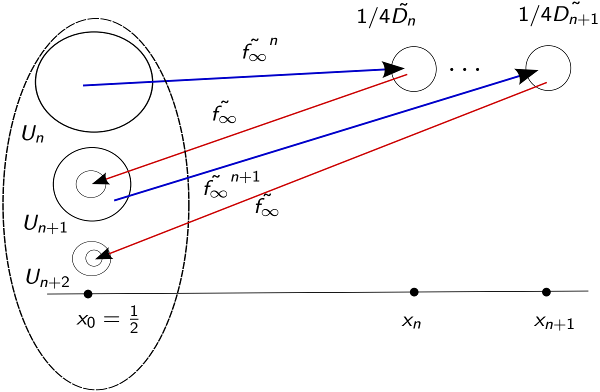

How he manage to do so...?

Bishop's QC-folding for entire maps in class $\mathcal{B}$

- We change g in the D_n's to be $g = \rho_n \rho(\tau^{d_n})$ so that $\rho_n(1/2) = w_{n+1}$ the center of U_{n+1}.
- The *new* dilatation is $O(r_{n+1})$ and it is supported in $O((d_n)^{-1})$ around the $\partial \tilde{D}_n$. Notice that the distance from $1/2$ to w_{n+1} is governed by r_{n+1}.
- The correction on ϕ is close to the identity in the whole plane but we only need to do so in unit discs centered at the points $\{x_0, \dots x_n\}$ so that U_{n+1} still is mapped to $\tilde{D}_{n+1}$ (if necessary, we increase d_n).
- We do these slightly modifications to be summable over n.

Thank you!!