On the Hausdorff dimension of the set of escaping parameters and the escaping set

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Transcendental entire functions

A. Eremenko -1986 proved that for transcendental entire functions the escaping set

$$I(f) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}$$

is non-empty and $J(f) = \partial I(f)$.

Transcendental meromorphic functions

P. Domínguez- 1998 extended Eremenko's result to transcendental meromorphic functions with poles.

Transcendental entire functions

A. Eremenko and M.Lyubich -1992 showed that if additionally an entire function f is in the class $\mathcal{B}$, then

$$I(f) \subset J(f).$$

Transcendental meromorphic functions

G. Stallard and P. Rippon- 1999 proved the same for transcendental meromorphic functions with poles i.e.

- if f is in the class $\mathcal{B}$, then $I(f) \subset J(f)$, which follows that $\mathrm{Int} I(f) = \emptyset$.
- Actually they proved that $I_R(f) \subset J(f)$ if R is sufficiently large, where $I_R(f) = \{z \in \mathbb{C} : \liminf_{n\to\infty} |f^n(z)| \geq R\}$. Note that $I(f) = \bigcap_{R>0} I_R(f)$.

Part I - The Hausdorff dimension of $I(f)$ - the upper bound

Theorem 1 - Urbański and K.- 2003

Let f be a non-constant elliptic function. Then

$$\dim_H(I(f)) \leq \frac{2q}{q+1},$$

where q is the maximal multiplicity of poles of f.

Theorem 2 - Bergweiler and K.- 2012

- Let $f \in \mathcal{B}$ be a transcendental meromorphic function
- $\rho = \rho(f) < \infty$, ∞ is not an asymptotic value
- there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except possibly finitely many, is at most M. Then

$$\mathrm{HD}(I(f)) \leq \frac{2M\rho}{2+M\rho} \quad \text{and} \quad \lim_{R\to\infty} \mathrm{HD}(I_R(f)) \leq \frac{2M\rho}{2+M\rho}.$$

For elliptic function ρ=2, so Th.2 implies Th.1.

Part I - The Hausdorff dimension of $I(f)$ - the lower bound

Theorem - K. -1995

Let f be a non-constant elliptic function. If the closure of the postcritical set is disjoint from the set of poles, then

$$\dim_H(I(f)) \geq \frac{2q}{q+1},$$

where q is the maximal multiplicity of poles of f.
Now one can prove that this assumption can be omitted

Corollary

Let f be a non-constant elliptic function. Then

$$\dim_H(I(f)) = \frac{2q}{q+1},$$

where q is the maximal multiplicity of poles of f.

Part II(a) - Structure of the set of repelling periodic points

Rational functions

Let f be a rational map. Then the set $Per_n(f) = \{z \in \mathbb{C} : \text{ z is a repelling periodic point of period n } \}$ is finite. So it has no accumulation points.

Transcendental entire functions (Baker, Bergweiler)

An entire transcendental function f has infinitely many repelling periodic points of period n for all $n \geq 2$ and ∞ is the unique accumulation point of $Per_n(f)$.

Transcendental non-entire functions (Baker-K.-Lü.)

Suppose that f is a transcendental non-entire function(*). Then $J(f) = \overline{O^-(\infty)}$. Let $z_1, z_2, \ldots, z_5 \in O^-(\infty) \setminus \{\infty\}$ are distinct. Define n_j by $f^{n_j}(z_j) = \infty$. Then there exists $j \in \{1, \ldots, 5\}$ such that z_j is a limit point of repelling periodic points of minimal period $n_j + 1$. Their multipliers tends to ∞.

Part II(b)- The role of complex analysis in complex dynamics

Comments

- N. Baker proved density of repelling periodic points in the Julia set of transcendental entire functions applying Ahlfors' theorem called *five island theorem*. Its statement was presented at Hinkkanen's lecture.
- There is another version of Five Islands Theorem used by Baker-K.-Lü to prove density of reppeling period points in the Julia of transcendental meromorphic functions (with poles).

Part II(b)- The role of complex analysis in complex dynamics

Ahlfors' Theorem

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function, and let $D_1, D_2, \ldots D_5$ be five simply connected domain in $\mathbb{C}$ with disjoint closure. Then

- there exists $j \in \{1, 2, \ldots, 5\}$
- and for any $R > 0$ there is a simply connected domain $G \subset \{z \in \mathbb{C} : |z| > R\}$

such f is a conformal map of G onto D_j.
If f has only finitely many poles, then 'five' may be replaced by 'three'.

Corollary

This version of Ahlfors' Five Island Theorem implies a very rich structure of repelling periodic points in $J(f) = \overline{O^{-}(\infty)}$ which I like to call double density of sources i.e. if $z \in J(f)$, then

- there is a sequence of prepoles $z_k \in f^{-n_k}(\infty)$ such that $z_k \to z$ and $n_k \to \infty$,
- for every z_k there is a sequence repelling priodic points period $n_k + 1$ which accumulate on z_k and their multipliers tend to ∞.

Please remember and distinguish 'density' of rep. periodic pts for meromorphic maps form 'density' of repelling periodic pts for entire maps!

Part II(a) - Structure of the set of repelling periodic points-revisited

We don't need Alfhors' theory to prove the following property.

Remark

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a non constant elliptic function. Then

- $J(f) = \overline{O^{-}(\infty)}$
- For every $n \in \mathbb{N}$ and every $z \in f^{-n}(\infty)$ there is a sequence of repelling periodic points $\{z_j\}_{j \in \mathbb{N}}$ of minimal period $n+1$ such that $z = \lim_{j \to \infty} z_j$. Their multipliers tends to ∞.

We have plenty of poles and a huge collection of reppelling periodic points.

Part II(c) - The Hausdorff dimension of the closure of the set of repelling periodic points

Theorem - Urbański and K.- 2003

Let f be a non-constant elliptic function, $Per_{2n}(f)$ - be the set of repelling periodic points of even period $2n$. Then

$$\dim_H(\overline{\bigcup_{n=1}^{\infty} Per_{2n}(f)}) > \frac{2q}{q+1}.$$

Consequently

$$\dim_H(J(f)) > \frac{2q}{q+1} > \dim_H(I(f)),$$

where q is the maximal multiplicity of poles of f.

The proof is based on the termodynamical formalism of IFS developed by Mauldin anf Urbański.

Part III- Escaping paramaters - 'brief' introduction

Basic definitions

For $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$, a lattice $\Lambda \subset \mathbb{C}$ is defined by $\Lambda = [\lambda_1, \lambda_2] = \{l\lambda_1 + m\lambda_2,\ l, m \in \mathbb{Z}\}$. Equivalently one may write $\Lambda = [1, \lambda_1]$, where $\mathrm{Im}\lambda_1 \neq 0$.

$$\mathcal{R} = \{t_1 + t_2\lambda_1;\ 0 \leq t_1, t_2 < 1\}$$

is the fundamental parallelogram of Λ.

Basic definitions

An elliptic function is a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ which is periodic with respect to a lattice Λ, i.e. $f(z) = f(z + l + m\lambda_1)$ for all $z \in \mathbb{C}$ and $l, m \in \mathbb{Z}$.

Part III - 'brief' introduction

The Weierstrass function $\wp_\Lambda$

The Weiestrass elliptic functions are defined by

$$\wp_\Lambda = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

for all $z \in \mathbb{C}$ and every lattice $\Lambda = [\lambda_1, \lambda_2]$, $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$.

Properties of $\wp_\Lambda$

- The Weierstrass function $\wp_\Lambda$ has pole of order 2 at lattice points
- Its derivative $\wp'_\Lambda$ has poles of order 3. It implies that $\wp_\Lambda$ has at most 3 critical points in the fundamendal parallelogram.

Properties of $\wp_\Lambda$

- In the fundamental parallelogram the $\wp_\Lambda$ has tree critical points $c_1 = \frac{\lambda_1}{2}$, $c_2 = \frac{\lambda_2}{2}$, $c_3 = \frac{\lambda_1+\lambda_1}{2}$, where $\Lambda = [\lambda_1, \lambda_2]$, $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$.
- $\wp_\Lambda$ no asymptotic values since $\wp_\Lambda$ is periodic.
- Thus $\wp_\Lambda$ is in the class $\mathcal{S}$.

Assumptions

Let $\wp_\Lambda : \mathbb{C} \to \overline{\mathbb{C}}$ be the Weierstrass elliptic function such that

- Λ is a triangular lattice i.e. $\lambda_2 = e^{2\pi i/3}\lambda_1 \Leftrightarrow \Lambda = e^{2\pi i/3}\Lambda$.
- the three critical values e_1, e_2 and e_3 of $\wp_\Lambda$ are the poles.

Remarks

The examples of such functions were given by Hawkins and Koss.

Assumptions

- We consider the one-parameter family of functions

$$f_\beta(z) = \beta\wp_\Lambda(z), \;\; \beta \in B(1, r), r > 0.$$

- The functions f_β are periodic and their critical points are the same as for the elliptic function f, while the critical values depend on the paramater β.
- For $\beta \in B(1, r)$ the orbits of critical points c_1, c_2 and c_3 under iterates of f_β behave symmetrically.

Part IV- Perturbations of the Weierstrass function $\wp_\Lambda$

Definition

As a counterpart of escaping set $I(f_\beta)$ we consider the set of escaping parameters in the family f_β, i.e.

$$\mathcal{E} = \{\beta \in B(1, r) \colon \lim_{n\to\infty} f^n_\beta(c_i) = \infty\}$$

where $c_1 = \frac{\lambda_1}{2}$, $c_2 = \frac{\lambda_2}{2}$, $c_3 = \frac{\lambda_1+\lambda_1}{2}$ are the critical points of f (resp. f_β).

Theorem - Gałazka

$$\dim_H(\mathcal{E}) \geq 4/3 = \dim_H(I(f_\beta)),$$

where $\mathcal{E}$ is the set of escaping parameters.

P. Gałazka, Math. Proc. Cambridge Phil. Soc. 154 (2013), pp. 97-118.

Motivation

- The dynamics and metric properties of the elliptic functions were investigated in series of papers written by: J. Hawkins, L. Koss, J. K., V. Mayer, M. Urbański....
- The class $\mathcal{E} \subset \{\beta\wp_\Lambda : \beta \in B(1, r),\ r > 0\}$ is an example of so called *critically tame elliptic functions* considered by M. Urbański and K. It is a generalization of non-recurrent elliptic functions.
- Estimate a measure/dimension of the critically tame elliptic maps.
- Compare the estimate of the Hausdorff dimension of escaping parameters and the Hausdorff dimension of escaping set.

Part V(b)- Critically tame elliptic functions

Definition

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be an elliptic function and $c \in \mathrm{Crit}(f)$. We say that f is critically tame if the following conditions are satisfied:

- if $c \in F(f)$, then there exists an attracting or parabolic cycle of period p, $S = \{z_0, f(z_0), ..., f^{p-1}(z_0)\}$ such that $\omega(c) = S$.
- if $c \in J(f)$, then one of the following holds:
  - $\omega(c)$ is a compact subset of $\mathbb{C}$ such that $c \notin \omega(c)$;
  - c is eventually mapped onto some pole;
  - $\lim_{n\to\infty} f^n(c) \to \infty$ and $\dim_H(J(f)) \geq \frac{2l_\infty}{l_\infty+1}$, where $l_\infty := \max\{p_c q_c\}$, q_c-multiplicity of the poles approached by critical values, p_c - the order or critical points escaping to ∞.

Part V(b) - Critically tame elliptic functions

Proposition

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic functions. Then f has neither Siegel discs and Herman rings nor Cremer points.

Theorem - Urbański and K.

Suppose that f is critically tame elliptic function, denote $h = \dim_H(J(f))$. Then there exist:

- a unique atomless h-conformal measure m for $f : J(f) \setminus \{\infty\} \to J(f)$ where m is ergodic, conservative and $m(Tr(f)) = 1$; $Tr(f) \subset J(f)$ denotes the set of all transitive points of f
- a non-atomic, σ-finite, ergodic, conservative and invariant measure μ for f, equivalent to the measure m. Additionally, μ is unique up to a multiplicative constant and is supported on $J(f)$.

Definition

- The measure μ is called *ergodic* if for $\forall G \in \mathcal{B}$ s.t. $f^{-1}(G) = G$ one has $\mu(G) = 0$ or $\mu(G^c) = 0$.
- The measure μ is called *conservative* if: $\forall G \in \mathcal{B}$ s.t. $\mu(G) > 0$ $\mu(\{z \in G : \sharp\{n \in \mathbb{N} : f^n(z) \in G\} < \infty, \}) = 0$.

Definition

$J_\mu(\infty)$ denotes the set of points of infinite condensation of the measure μ. A point $z \in J_\mu(\infty)$ if for any nbhd U of z, $\mu(U) = \infty$.

Remark

σ-finite measure μ is finite iff $J_\mu(\infty)$ is empty.

Lemma

$J(\infty) \subset \Omega(f) \cup \{\infty\}$, $\Omega(f)$-set of parabolic periodic points.

Theorem - Urbański and K.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function. If $h = \dim_H(J(f)) = 2$, then $J(f) = \overline{\mathbb{C}}$. If $h < 2$, then

1. $\mathrm{H}_s^h(J(f)) = 0$.
2. $\Pi_s^h(J(f)) > 0$.
3. $\Pi_s^h(J(f)) = \infty$ if and only if $\Omega(f) \neq \emptyset$, $\Omega(f)$ is the set of parabolic periodic points.

Part V(b) - Critically tame elliptic functions with finite invariant measure

Theorem - Urbański and K.

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame elliptic function whose Julia set is equal to the entire complex plane $\mathbb{C}$, then there exists a unique Borel probability f-invariant measure μ equivalent to the planar Lebesgue measure on $\mathbb{C}$.

Definition

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be critically tame elliptic functions, f has no parabolic periodic points and $\mathrm{Crit}_\infty(f) = \emptyset$ i.e. $J_\mu(\infty) = \emptyset$. We call these maps of finite character. They have a finite f-invariant measure μ_h, where $h = \dim_h(J(f))$.

Part V(b) - Critically tame elliptic functions with finite invariant measure-maps of finite type

Decay of correlation - Urbański and K.

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function of finite character and if μ is the corresponding probability f-invariant measure equivalent to the h-conformal measure m, then for the dynamical system (f, μ) the following. hold. Fix $\alpha \in (0, 1]$ and a bounded function $g : J(f) \to \mathbb{R}$ which is Hölder continuous with respect to the Euclidean metric on $J(f)$ with the exponent α. Then for every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that

$$\left| \int \psi \circ f^n \cdot g d\mu - \int g dmu \int \psi d\mu \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on α.

Part V(b) - Critically tame elliptic functions with finite invariant measure - maps of finite type

The Central Limit Theorem - Urbański and K.

The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$, i.e. for which there is no square integrable function η for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that
$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$ in distribution.

Part V(b) - Critically tame elliptic functions with finite invariant measure - maps of finite type

The Law of Iterated Logarithm - Urbański and K.

The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$. This means that there exists a real positive constant A_g such that such that μ_ϕ almost everywhere $\limsup_{n\to\infty} \frac{S_n g - n\int g d\mu}{\sqrt{n \log\log n}} = A_g$.

Theorem - Urbański -K.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be critically tame map of finite type. Then Sinay-Kolmogorov entropy of f with respect to a probability invariant measure exists and is finite.

Comments

- Let $\wp_\Lambda$ be the Weierstrass function defined by triangular lattice $\Lambda = [\lambda_1, \lambda_2]$ i.e. $\lambda_2 = e^{2\pi i/3}\lambda_1$.
- We assume that the critical values e_1, e_2, e_3 are the poles. $f_\beta := \beta\wp_\Lambda$, $\beta \in B(1, r)$.
- The maps corresponding to escaping parameters $\mathcal{E} = \{\beta \in B(1, r)\colon \lim_{n\to\infty} f_\beta^n(c_i) = \infty\}$, ($c_i, c_2, c_3$ the citical points) are critically tame since
  1. $J(f_\beta) = \mathbb{C} \Rightarrow \dim_H(J(f)) = 2$
  2. $l_\infty := \max\{\frac{2q_cp_c}{p_cq_c+1}\} = \frac{2x2x2}{2x2+1} = \frac{8}{5}\}$.
  3. $\dim_H(J(f_\beta)) = 2 > l_\infty = \frac{8}{5}$

Comments

- $\dim_H(I(f_\beta)) = \frac{2q}{q+1} = \frac{4}{3}$
- $\dim_H(\mathcal{E}) \geq \frac{4}{3}$.
- for $\beta \in \mathcal{E}$, f_β admits 2-conformal measure m for $f_\beta : J(f) \setminus \{\infty\} \to J(f_\beta)$, m is ergodic, conservative and $m(Tr(f_\beta)) = 1$. (*)
- there exists a non-atomic, finite, ergodic, conservative and invariant measure μ for f_β, equivalent to the measure m. Additionally, μ is unique up to a multilicative constant.
- the measure μ observes stochastics laws: Decay of Correlation, Central Limit Theorem, Law of Iterated Logarithm.
- f_β has a finite Sinay-Kolmogorov metric entropy with respect to a measure μ.

Assumptions

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function such that

i) one of its critical values, denoted by $f(c_1) \neq 0$, is a pole of the maximal multiplicity q and

ii) all the other critical values are attracted by attracting periodic points.

We consider the one-parameter family of functions $f_\beta(z) = \beta f(z)$, $\beta \in B(1, r)$, $r > 0$.

Remark

The functions f_β are periodic and their critical points are the same as for the elliptic function f, the critical values depend on the paramater β.

Part VI - Another class of elliptic functions

Definition

As a counterpart of escaping set $I(f_\beta)$ we consider the set of escaping parameters in the family f_β, i.e.
$\mathcal{E} = \{\beta \in B(1,r)\colon \lim_{n\to\infty} f_\beta^n(c_1) = \infty\}$, where $r > 0$, c_1 is a critical point of f (f_β).

Theorem (Gałazka-K.)

Let $f_\beta = \beta f$, $\beta \in \mathbb{C}$ be a one-parameter of family of elliptic functions such that

i) one of its critical values, denoted by $f(c_1) \neq 0$, is a pole of the maximal multiplicity q and

ii) all the other critical values are attracted by attracting periodic points.

Then $\dim_H(\mathcal{E}) = \frac{2q}{q+1}$, where $\mathcal{E}$ is the set of escaping parameters in the family f_β.

Example 1.

with $q = 2$

For the Weierstrass function $\wp_\Lambda$ generated by the lattice Λ with the invariants $g_2(\Lambda) \approx 26.5626$ and $g_3(\Lambda) \approx -26.2672$, Hawkins and Koss proved that

- $\wp_\Lambda$ has an attracting fixed point $p \approx 1.5566$, which attracts two critical values $e_1 \approx 1.5539$ and $e_3 \approx 1.4206$,
- while the third critical value $e_2 \approx -2.9746$ is a pole.
- We don't know if the maps in $\mathcal{E}$ are critically tame since we don't know the Hausdorff dimension of the Julia set $\dim_H(J(f))$
- The required condition says that $\dim_H(J(f)) > \max\{\frac{2q_c p_c}{q_c p_c + 1}\} = \frac{8}{5}$ since $p_c = 2$ and $q_c = 2$.
- For every elliptic functions $\dim_H(J(f)) > \frac{2q}{q+1} = \frac{4}{3}$. But this estimate is not good enough.

Example 2 with $q = 2$

- Let $\Gamma = [d + ai, d - di], d > 0$ be the standard rhombic square lattice, $\wp_\Gamma$ be the corresponding Weierstrass function. The critical points are $c_1 = (d + di)/2$, $c_1 = (d - di)/2$, $c_3 = d$ then $\wp_\Gamma(d + di) = e_1 = -di$.
- Define an elliptic function $h := \wp_\Gamma + di$. h is an elliptic functions with poles of order 2 at each lattice point of Γ. $h(di) = di$ and $h'(di) = 0$.
- $f_\beta = \beta h$ is an elliptic function, $\beta \in B(0, r)$
- f_β has an attracting periodic point and the first critical value $f_\beta(d)$ belongs to the basin of attraction of attracting periodic point.
- the second critical value $f_\beta(d + di)$ is a pole for every $\beta \in \mathbb{C} \setminus \{0\}$.
- the forward trajectory of the third critical value $f_\beta(d - di)$ tends to ∞.

cont. Example 2 with $q = 2$

To estimate the estimate $\dim_H(J(f))$ we apply the following lemma.

Lemma

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non constant elliptic function such that a crtical point c od order p_c is eventually mapped onto a pole of a multiplicity q then

$$\dim_H(J(f_\beta)) > \frac{2p_c q}{p_c q + 1}.$$

Collorary

- $2 > \dim_H(J(f_\beta)) > \frac{8}{5} > \frac{4}{3} = \dim_H(I(f))$
- $\dim_H(\mathcal{E}) = \frac{4}{3}$.
- The maps $f_\beta, \beta \in \mathcal{E}$, are critically tame.

Remarks

The perturbations of Weierstrass function $\wp_\Lambda$

The proof of the upper bound on $\dim_H(\mathcal{E})$ can be extended to the family $f_\beta = \beta\wp_\Lambda$ considered before. Thus $\dim_H(\mathcal{E}) = \frac{4}{3}$.

Simply periodic mermorhic functions

The proof of the Theorem can be extended to a one-parameter family $f_\beta = \beta f$, where f is a simply periodic meromorphic function in the class $\mathcal{S}$ satisfying similar assumptions as considered elliptic functions.i.e one critical/asymptotic value is a pole, the others are contained in a basin of atraction of an atracting periodic cycle. So we get

$$\dim_H(\mathcal{E}) = \dim_H(I(f)) = \frac{q}{q+1},$$

where q is the maximal multiplicity of poles.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Assumptions

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be an elliptic function such that:

- one critical value $f(c_1) \neq 0$ is a pole of the multiplicity q
- and all the other critical values $f(c_i); i = 2; \ldots, k$, are attracted by attracting periodic points.

Remark

If $f(c_1) = 0$ is a pole, then $f_\beta(c_1) = 0$ and $f_\beta^2(c_1) = \infty$ for every $\beta \in \mathbb{C} \setminus \{0\}$. In this case $\mathcal{E} = \emptyset$.

Assumptions

We consider the one-parameter family of functions
$f_\beta(z) = \beta f(z)$; $\beta \in B(1, r)$ *for* $0 < r < \frac{1}{4} - \frac{1}{2\alpha+4}$,
where $a = \sin(\pi/8)$.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Remark

As f has only finitely many periodic sinks and hyperbolicity preserves under perturbation, without loss of generality we may assume that for every $b \in B(1, r)$, its critical values, except of $f_\beta(c_1)$, belong to basins of attraction of the attracting periodic cycles.

The poles

From now on, we consider only poles of multiplicity q from the set $f_q^{-1}(\infty) = \{f(c_1) + k\lambda_1 + m\lambda_2 : k; m \in \mathbb{Z}\}$, $f(c_1)$ is a pole of f.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

The sequence of half-annuli

We consider a sequence of radii

$$R_k := a^{k-1} R_1, \quad k \geq 2,\ a > a_0 > 2$$

and a sequence of half-annuli for $k \geq 2$

$$P^+(0, R_k, 2R_k) := \{z \in \mathbb{C} \colon R_k < |z| < 2R_k,\ \phi < \arg z < \phi + \pi\}$$

The condition $a > a_0 \geq 2$ guarantees that the half-annuli are pairwise disjoint.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

The segments

For a pole

$$b \in f_q^{-1}(\infty) \cap P^+(0, R_k, 2R_k), \; R_k = aR_{k-1}$$

we define segment (part of the neighbourhood)

$$U(b, \varepsilon) := \{z \in \mathbb{C} \colon -\frac{3\pi}{4q} \leq \operatorname{Arg}(z-b) \leq \frac{3\pi}{4q}, \; |z-b| \leq \varepsilon\}$$

where

- $f_q^{-1}(\infty) = \{f(c_1) + k\lambda_1 + m\lambda_2 : \; k; m \in \mathbb{Z}\}$, $f(c_1)$ is a pole of f
- and $P^+(0, R_k, 2R_k) := \{z \in \mathbb{C} \colon R_k < |z| < 2R_k, \; \phi < \arg z < \phi + \pi\}$

Figure 1

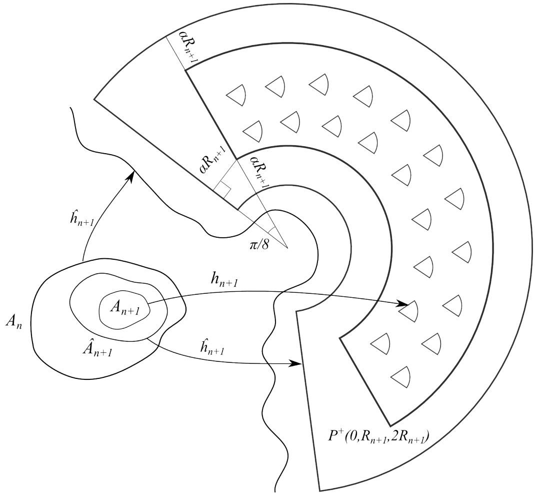


Figure :

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Remark

For all poles $b \in f_q^{-1}(\infty)$ and $\beta \in B(1,r)$.

$$\{z \in \mathbb{C} : |z| > R_2 \geq \tilde{R}_2,\ \phi \leq \arg z \leq \phi + \pi\} \subset f_\beta(U(b,\varepsilon))$$

Definition

We consider auxiliary functions $h_n(\beta) = f_\beta^n(c_1)$, $n \in \mathbb{N}$, which are defined outside a countable set of parameters.

We want to define a sequence of covers of the set of escaping parameters in the family f_β, i.e.

$$\mathcal{E} = \{\beta \in B(1,r) \colon \lim_{n\to\infty} f_\beta^n(c_1) = \infty\}$$

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

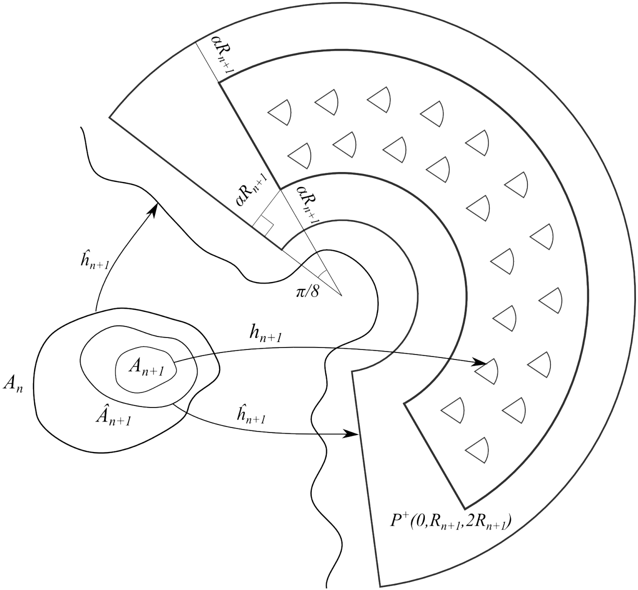
Definition

We define the following family of sets for a given $a > a_0$.

$$\begin{aligned}
\mathcal{A}_0(a) &= \{A_0 = B(1,r)\},\\
\mathcal{A}_1(a) &= \{A_1 = h_1^{-1}(U(f(c_1),\varepsilon)) \subset A_0\},\\
\mathcal{A}_2(a) &= \{A_2 \subset A_1 \mid \exists b^{(2)} \in f_q^{-1}(\infty)\colon\\
&\qquad U(b^{(2)},\varepsilon) \subset P^+(0,R_2,2R_2),\\
&\qquad A_2 \text{ is a component of } h_2^{-1}(U(b^{(2)},\varepsilon))\},\\
&\dots\\
\mathcal{A}_k(a) &= \{A_k \subset A_{k-1} \mid \exists b^{(k)} \in f_q^{-1}(\infty)\colon\\
&\qquad U(b^{(k)},\varepsilon) \subset P^+(0,R_k,2R_k),\\
&\qquad A_k \text{ is a component of } h_n^{-1}(U(b^{(k)},\varepsilon))\}, k>1
\end{aligned}$$

Let $\mathcal{U}_k(a) = \bigcup_{A_k \in \mathcal{A}_k(a)} A_k, \quad A(a) = \bigcap_{k=1}^{\infty} \mathcal{U}_k(a)$.

Figure 1



The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Remark

For each $n \in \mathbb{N}$, the set $\mathcal{A}_n(a)$ defined above is non-empty.

Proposition

There is $a_0 > 2$ such that for every $a > a_0$ there is a Cantor subset $A(a)$ of $\mathcal{E}$ and for this subset

$$\dim_H(A(a)) \geq \frac{2q}{q+1} - \frac{6 \log 2}{\log a}.$$

Corollary

For $a \nearrow \infty$ we have $\frac{2q}{(q+1)} - 6\frac{\log 2}{\log a} \nearrow \frac{2q}{(q+1)}$ and $\dim_H(\mathcal{E}) \geq \frac{2q}{(q+1)}$.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Mc Mullen's Lemma

For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a finite collection of disjoint compact subsets of $\mathbb{R}^d$, each of which has positive d-dimensional Lebesgue measure.

- Define $\mathcal{U}_n = \bigcup_{A_n \in \mathcal{A}_n} A_n$ and $A = \bigcap_{n=1}^{\infty} \mathcal{U}_n$.
- Suppose that for each $A_n \in \mathcal{A}_n$ there is $A_{n+1} \in \mathcal{A}_{n+1}$ and a unique $A_{n-1} \in \mathcal{A}_{n-1}$ such that $A_{n+1} \subset A_n \subset A_{n-1}$.
- If Δ_n, d_n are such that
  - for each $A_n \in \mathcal{A}_n$, $\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geq \Delta_n > 0$,
  - $\mathrm{diam}(A_n) \leq d_n < 1$ and $d_n \overset{n\to\infty}{\longrightarrow} 0$,

then $\dim_H(A) \geq d - \lim_{n\to\infty} \sum_{j=1}^{n} \frac{|\log \Delta_j|}{|\log d_n|}$.

The proof - $\dim_H(\mathcal{E}) \geq \frac{2q}{q+1}$

Remark

For each $A_n \in \mathcal{A}_n$, $n \geq 1$, the map h_n is conformal on A_n, where $h_n(\beta) = f_\beta^n(c_1)$, $n \in \mathbb{N}$.

Remark

We showed in fact that

- the segments $U(b_n, \varepsilon) \subset P^+(0, R_n, 2R_n)$, $n \geq 2$, are in one-to-one correspondence with the sets $A_n \subset \mathcal{U}_n \cap A_{n-1}$ for each $A_{n-1} \in \mathcal{A}_{n-1}$.
- Hence, each $\mathcal{A}_n$, $n \geq 1$, is a finite collection of the sets A_n.

Lemma

Let $A_n \in \mathcal{A}_n$, $n \geq 2$. Then for every $\beta \in A_n$

$$h_n'(\beta) = \frac{1}{\beta} \prod_{k=1}^{n-1} f_\beta'(f_\beta^k(c_1)) \left[f_\beta(c_1) + \sum_{k=2}^{n} \frac{f_\beta^k(c_1)}{\prod_{i=1}^{k-1} f_\beta'(f_\beta^i(c_1))} \right].$$

where $h_n(\beta) = f_\beta^n(c_1)$, $n \in \mathbb{N}$.

Remark

There are universal constants $C_1 > 0$, $C_2 > 0$ such that

$$|f_\beta(z)| \asymp \frac{C_1}{|z-b|^q}, \quad |f_\beta'(z)| \asymp \frac{C_2}{|z-b|^{q+1}}$$

for each $b \in f_q^{-1}(\infty)$, every $z \in B(b, \varepsilon)$ and all $\beta \in B(1, r)$.

Lemma

Let $A_n \in \mathcal{A}_n$, $n \geq 2$. Then for every $\beta \in A_n$

$$\frac{1}{2(1+r)} \left(\frac{C_2}{C_1^{\frac{q+1}{q}}} \right)^{n-1} a^{\frac{(q+1)n(n-1)}{2q}} R_1^{\frac{(q+1)n-1}{q}}$$

$$\leq \left| h_n'(\beta) \right| \leq$$

$$\frac{5}{2(1-r)} \left(\frac{2^{\frac{q+1}{q}} C_2}{C_1^{\frac{q+1}{q}}} \right)^{n-1} a^{\frac{(q+1)n(n-1)}{2q}} R_1^{\frac{(q+1)n-1}{q}}.$$

where $R_k = a^{k-1} R_1$, $a > a_0$, q- is the multiplicity of q, $\beta \in B(1, r)$.

Final estimates

$$\dim_H(A) \geq d - \lim_{n\to\infty} \sum_{j=1}^{n} \frac{|\log \Delta_j|}{|\log d_n|},$$

where $\frac{\mathrm{vol}(\mathcal{U}_{n+1} \cap A_n)}{\mathrm{vol}(A_n)} \geq \Delta_n > 0$, and

$$\mathrm{diam}(A_n) \leq d_n \stackrel{n\to\infty}{\longrightarrow} 0,$$

$$\sum_{j=1}^{n} |\log \Delta_j| \sim \frac{3(q+1)(n+2)(n-1)}{q} \log 2 + \frac{n(n+1)}{q} \log a$$

$$|\log d_n| \sim \frac{(q+1)n(n-1)}{2q} \log a + \frac{(q+1)n-1}{q} \log R_1$$

Final estimate

$$\dim_H(A(a))$$

$$\geq 2 - \limsup_{n\to\infty} \frac{\frac{3(q+1)(n+2)(n-1)}{q}\log 2 + \frac{n(n+1)}{q}\log a}{\frac{(q+1)n(n-1)}{2q}\log a + \frac{(q+1)n-1}{q}\log R_1}$$

$$= 2 - \frac{\frac{1}{q}\log a + \frac{3(q+1)}{q}\log 2}{\frac{q+1}{2q}\log a}$$

$$= 2 - \frac{2}{q+1} - \frac{6\log 2}{\log a} = \frac{2q}{q+1} - \frac{6\log 2}{\log a}.$$

The proof - $\dim_H(\mathcal{E}) \leq \frac{2q}{q+1}$

The poles

From now on, we consider only poles of multiplicity q from the set $f_q^{-1}(\infty) = \{f(c_1) + k\lambda_1 + m\lambda_2 : \ k; m \in \mathbb{Z}\}$.

The cover of poles

For the poles in $f_q^{-1}(\infty) \cap \{z \in \mathbb{C} : \ |z| > R\}$
we define the family of disjoint balls $B(b, 2\varepsilon)$, ε is small.

Definition

As previously, we consider the auxiliary functions

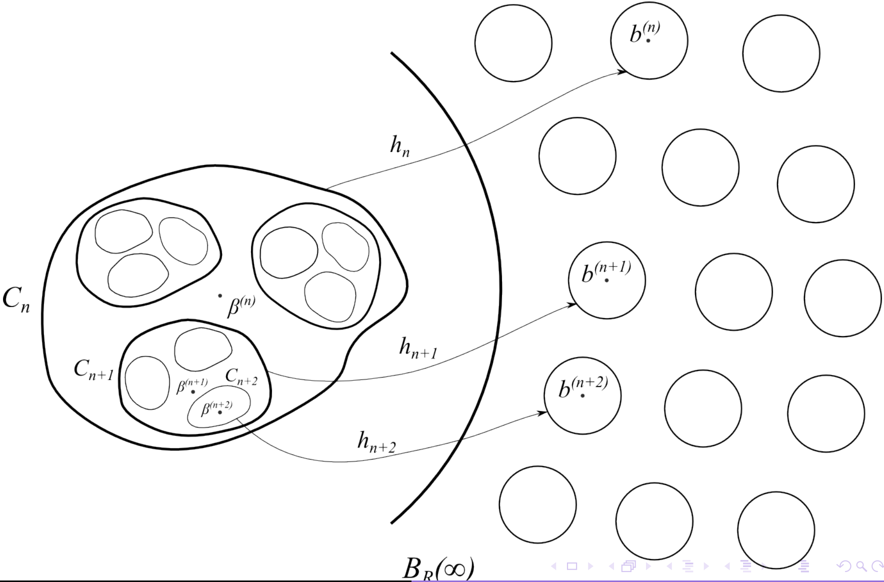
$$h_n(\beta) = f_\beta^n(c_1), \ n \geq 1, \ \beta \in B(1, r)$$

Definition

We define the following family of sets.

$$
\begin{aligned}
&\mathcal{C}_0 = \{C_0 = B(1,r)\},\\
&\mathcal{C}_1 = \{C_1 = h_1^{-1}(B(f(c_1),\varepsilon)) \subset C_0\},\\
&\mathcal{C}_2 = \{C_2 \subset C_1 \mid \exists b^{(2)} \in f_q^{-1}(\infty) \colon B(b^{(2)},2\varepsilon) \subset B_R(\infty),\\
&\qquad C_2 \text{ is a component of } h_2^{-1}(B(b^{(2)},\varepsilon))\},\\
&\quad \dots\\
&\mathcal{C}_n = \{C_n \subset C_{n-1} \mid \exists b^{(n)} \in f_q^{-1}(\infty) \colon B(b^{(n)},2\varepsilon) \subset B_R(\infty),\\
&\qquad C_n \text{ is a component of } h_n^{-1}(B(b^{(n)},\varepsilon))\},\\
&\quad \dots
\end{aligned}
$$

Figure 2



The proof - $\dim_H(\mathcal{E}) \le \frac{2q}{q+1}$

Definition

We define the set of parameters

$$\mathcal{E}_R = \{\beta \in B(1,r)\colon\ \forall n \ge 2\ |f_\beta^n(c_1)| > R\}.$$

Then $\mathcal{E} \subset \mathcal{E}_R$ where $\mathcal{E} = \{\beta \in B(1,r)\colon\ \lim_{n\to\infty} f_\beta^n(c_1) = \infty\}$, $r > 0$, c_1 is a critical point of f (f_β).

Theorem

Suppose $\mathcal{E}_R$ is the set defined above. Then

$$\dim_H(\mathcal{E}_R) \le \frac{2q}{q+1}.$$

and $\dim_H(\mathcal{E}) \le \dim_H(\mathcal{E}_R) \le \frac{2q}{q+1}$.

The proof - $\dim_H(\mathcal{E}) \le \frac{2q}{q+1}$

The family $\mathcal{C}_n$ covers $\mathcal{E}_R$.

- C_n is a component of $h_n^{-1}(B(b^{(n)}, \varepsilon))$.
- For $n \ge 2$, consider a sequence of sets $C_n \subset C_{n-1} \subset \ldots \subset C_2 \subset C_1$, $C_k \in \mathcal{C}_k$, $k = 1, \ldots n$, and the corresponding sequence of poles $b^{(n)}, b^{(n-1)}, \ldots b^{(2)}, b^{(1)} = f(c_1)$.
- Taking $\beta \in C_n$ and $z = h_{n-1}(\beta) = f_\beta^{n-1}(c_1)$, we have $|b^{(n)}| \asymp |f(z)| \asymp \frac{1}{|z - b^{(n-1)}|^q}$ and $|f'(z)| \asymp |b^{(n)}|^{\frac{q+1}{q}}$.

The proof - $\dim_H(\mathcal{E}) \leq \frac{2q}{q+1}$

- Since $|b^{(n)}| > R > 1$, it follows that
$$\begin{aligned}|h_n'(\beta)| &= |(\beta f(h_{n-1}(\beta))'| \\ &= |f(h_{n-1}(\beta)) + \beta f'(h_{n-1}(\beta))\, h_{n-1}'(\beta)| \\ &\asymp |b^{(n)}|^{\frac{q+1}{q}} |h_{n-1}'(\beta)|.\end{aligned}$$
- and, by induction,
$$|h_n'(\beta)| \asymp |b^{(n)}|^{\frac{q+1}{q}} |b^{(n-1)}|^{\frac{q+1}{q}} \dots |b^{(2)}|^{\frac{q+1}{q}},$$
as $|h_1'(\beta)| = |f(c_1)|$.
- For $\beta \in C_{n-1} \setminus \{\beta^{(n-1)}\}$ with $h_{n-1}(\beta^{(n-1)}) = b^{(n-1)}$, the estimate
$$|h_n'(\beta)| \asymp \frac{1}{|h_{n-1}(\beta) - b^{(n-1)}|^{q+1}}$$
follows that h_n does not have critical values in $B_R(\infty) = \{z \in \mathbb{C}; |z| > r\}$ (shrinking ε if necessary).

The family $\mathcal{C}_n$ covers $\mathcal{E}_R$

$$\mathrm{diam}(C_n) \le \varepsilon K |(h_{n,i}^{-1})'(b^{(n)})| \asymp \frac{\varepsilon K}{\left|b^{(n)}\right|^{\frac{q+1}{q}} \left|b^{(n-1)}\right|^{\frac{q+1}{q}} \dots \left|b^{(2)}\right|^{\frac{q+1}{q}}}. \quad (1)$$

In the following part we use the notation $\left|b^{(k)}\right| > R$, considering poles $b^{(k)} \in f_q^{-1}(\infty) \cap B_R(\infty)$. Taking $s > 0$ and using (1), we get

$$\sum_{C_n \in \mathcal{C}_n} \operatorname{diam}^s(C_n)$$

$$\leq q^{n-1} \sum_{|b^{(2)}|>R} \sum_{|b^{(3)}|>R} \cdots \sum_{|b^{(n)}|>R} \frac{\varepsilon^s K^s}{|b^{(n)}|^{\frac{q+1}{q}s} |b^{(n-1)}|^{\frac{q+1}{q}s} \cdots |b^{(2)}|^{\frac{q+1}{q}s}}$$

$$= \varepsilon^s K^s \left(\sum_{|b|>R} \frac{q}{|b|^{\frac{q+1}{q}s}} \right)^{n-1} .$$

Since the series $\sum_{b \in f_q^{-1}(\infty)} \frac{1}{|b|^m}$ converges for all $m > 2$, then there exists $\tilde{R} > 0$ such that

$$\sum_{b \in f_q^{-1}(\infty) \cap \{z \in \mathbb{C};\, |z|>R\}} \frac{q}{|b|^{(q+1)s/q}} < 1$$

for all $R \geq \tilde{R}$ and $s > 2q/(q+1)$.

- It follows that $\sum_{C_n \in \mathcal{C}_n} \mathrm{diam}^s(C_n) < \varepsilon^s K^s$ and diameters of the sets from $\mathcal{C}_n$, which cover $\mathcal{E}_R$, converge uniformly to 0 when $n \to \infty$.
- Hence, the s-dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{E}_R) \le \varepsilon^s K^s < +\infty$ and
- $\dim_H(\mathcal{E}_R) \le s$. Letting $s \searrow 2q/(q+1)$, we obtain

$$\dim_H(\mathcal{E}_R) \le 2q/(q+1).$$

The role of complex analysis in ergodic theory of transcendetal dynamics

Theorem Swiatek-K.

(Z1) Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function with finitely many singular values (f is of class $\mathcal{S}$). Suppose further that all poles of f have multiplicities bounded by M.

(Z2) $J(f) = \overline{C}$ and f satisfies the stronger Misiurewicz condition i.e. $P(f) \subset B(0, R)$ and $\overline{P(f)} \cap Crit(f) = \emptyset$.

(Z3)

$$\int_{r_0}^{\infty} \frac{T(r, f)}{r^{1+\frac{2}{M}}} dr < \infty,$$

where $a_j, j = 1, \ldots, n$, are the asymptotic values of f, $r_0 > 0$.

Then, f has an ergodic, probabilistic invariant measure which is absolutely continuous with respect to the Lebesgue measure.