An introduction to Nevanlinna theory

Jim Langley, ICMS, May 2013

May 17, 2013

1. Introduction

- Two key formulas:

$$\begin{aligned} T(r) &= m(r,a) + N(r,a) + O(1) \\ \sum_{(a)} \delta(a) &\leq 2. \end{aligned}$$

1. Introduction

- Two key formulas:

$$\begin{aligned} T(r) &= m(r,a) + N(r,a) + O(1) \\ \sum_{(a)} \delta(a) &\leq 2. \end{aligned}$$

- The first part of this talk is a sketch of Nevanlinna's first and second fundamental theorems and the defect relation.
  The second part will sketch some results of Teichmüller.

1. Introduction

- Two key formulas:

$$\begin{aligned} T(r) &= m(r,a) + N(r,a) + O(1) \\ \sum_{(a)} \delta(a) &\leq 2. \end{aligned}$$

- The first part of this talk is a sketch of Nevanlinna's first and second fundamental theorems and the defect relation.
  The second part will sketch some results of Teichmüller.
- All functions f in this talk are assumed non-constant and meromorphic in the plane.

1. Introduction

- Two key formulas:

$$\begin{aligned} T(r) &= m(r,a) + N(r,a) + O(1) \\ \sum_{(a)} \delta(a) &\leq 2. \end{aligned}$$

- The first part of this talk is a sketch of Nevanlinna's first and second fundamental theorems and the defect relation.
  The second part will sketch some results of Teichmüller.
- All functions f in this talk are assumed non-constant and meromorphic in the plane.
- **Caveat:** some formulas need to be adjusted if $f(0) = 0, \infty$ ("This tiresome modification is one of the minor irritations of the theory"- W.K. Hayman).

2. The Nevanlinna functionals

- Let f be meromorphic, with no zeros/poles on $|z| = R > 0$.

2. The Nevanlinna functionals

- Let f be meromorphic, with no zeros/poles on $|z| = R > 0$.
- Let the zeros and poles of f in $0 < |z| < R$ be $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, repeated according to multiplicity. Set

$$g(w) = f(w) \prod_{j=1}^{m} \left(\frac{R(w - a_j)}{R^2 - \overline{a_j} w} \right)^{-1} \prod_{k=1}^{n} \left(\frac{R(w - b_k)}{R^2 - \overline{b_k} w} \right),$$

Then $|g(w)| = |f(w)|$ for $|w| = R$, and $\log |g|$ is harmonic on $|w| \leq R$.

2. The Nevanlinna functionals

- Let f be meromorphic, with no zeros/poles on $|z| = R > 0$.
- Let the zeros and poles of f in $0 < |z| < R$ be $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, repeated according to multiplicity. Set

$$g(w) = f(w) \prod_{j=1}^{m} \left(\frac{R(w - a_j)}{R^2 - \overline{a_j} w} \right)^{-1} \prod_{k=1}^{n} \left(\frac{R(w - b_k)}{R^2 - \overline{b_k} w} \right),$$

  Then $|g(w)| = |f(w)|$ for $|w| = R$, and $\log |g|$ is harmonic on $|w| \leq R$.
- So Poisson's formula yields, for $z = re^{i\theta}$, $|z| = r < R$,

$$\log |g(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\phi})| \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi.$$

2. The Nevanlinna functionals

- Let f be meromorphic, with no zeros/poles on $|z| = R > 0$.
- Let the zeros and poles of f in $0 < |z| < R$ be $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, repeated according to multiplicity. Set

$$g(w) = f(w) \prod_{j=1}^{m} \left(\frac{R(w - a_j)}{R^2 - \overline{a_j} w} \right)^{-1} \prod_{k=1}^{n} \left(\frac{R(w - b_k)}{R^2 - \overline{b_k} w} \right),$$

  Then $|g(w)| = |f(w)|$ for $|w| = R$, and $\log |g|$ is harmonic on $|w| \leq R$.
- So Poisson's formula yields, for $z = re^{i\theta}$, $|z| = r < R$,

$$\log |g(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\phi})| \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi.$$

- Alternatively, apply Cauchy's integral formula to

$$V(w) = \left(\frac{R^2 - |z|^2}{R^2 - \overline{z}w} \right) \log g(w).$$

- Taking $z = 0$ gives

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\phi})|\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\,d\phi$$

(still valid even if f has zeros/poles on $|z| = R$).

2. The Nevanlinna functionals

- Taking $z = 0$ gives

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\phi})| \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \, d\phi$$

(still valid even if f has zeros/poles on $|z| = R$).

- In particular we get *Jensen's formula*

$$\begin{aligned} \log |f(0)| \quad = \quad & \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \, d\phi \\ & - \sum_{j=1}^{m} \log \frac{R}{|a_j|} + \sum_{k=1}^{n} \log \frac{R}{|b_k|}. \end{aligned}$$

2. The Nevanlinna functionals

- Nevanlinna writes

$$\log^+ x = \max\{\log x, 0\}, \quad \log x = \log^+ x - \log^+ \frac{1}{x} \quad (x > 0),$$

2. The Nevanlinna functionals

- Nevanlinna writes

$$\log^+ x = \max\{\log x, 0\}, \quad \log x = \log^+ x - \log^+ \frac{1}{x} \quad (x > 0),$$

- and

$$\begin{aligned} m(R,f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| \, d\phi, \\ N(R,f) &= \sum_{k=1}^{n} \log \frac{R}{|b_k|} = \int_0^R \frac{n(t,f)}{t} \, dt, \end{aligned}$$

using integration by parts, where

$n(t,f) =$ number of poles of f, counting multiplicity, in $|z| \leq t$.

2. The Nevanlinna functionals

- Next,

$$T(R,f) = m(R,f) + N(R,f)$$

(characteristic = proximity function + counting function)

2. The Nevanlinna functionals

- Next,
$$T(R,f) = m(R,f) + N(R,f)$$
(characteristic = proximity function + counting function)
- and Jensen's formula becomes
$$\log|f(0)| = T(R,f) - T(R,1/f) \quad (f(0) \neq 0, \infty).$$

2. The Nevanlinna functionals

- Next,

$$T(R,f) = m(R,f) + N(R,f)$$

(characteristic = proximity function + counting function)

- and Jensen's formula becomes

$$\log|f(0)| = T(R,f) - T(R,1/f) \quad (f(0) \neq 0, \infty).$$

- Among the easiest examples is

$$T(R,\exp) = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} R\cos\phi\, d\phi = \frac{R}{\pi} = \frac{\log M(R,\exp)}{\pi}.$$

2. The Nevanlinna functionals

- Next,

$$T(R,f) = m(R,f) + N(R,f)$$

(characteristic = proximity function + counting function)

- and Jensen's formula becomes

$$\log|f(0)| = T(R,f) - T(R,1/f) \quad (f(0) \neq 0, \infty).$$

- Among the easiest examples is

$$T(R,\exp) = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} R\cos\phi\,d\phi = \frac{R}{\pi} = \frac{\log M(R,\exp)}{\pi}.$$

- For entire f, the growth of $T(R,f) = m(R,f)$ is roughly comparable to that of $\log M(R,f)$.

3. Rates of growth

- For all non-constant meromorphic f, the characteristic $T(R, f)$ is increasing (Cartan's formula).

3. Rates of growth

- For all non-constant meromorphic f, the characteristic $T(R, f)$ is increasing (Cartan's formula).
- If f is rational with $f = P/Q$, $\deg P = p$, $\deg Q = q$, then

$$T(R, f) = \max\{p, q\} \log R + O(1) \quad \text{as} \quad R \to \infty.$$

3. Rates of growth

- For all non-constant meromorphic f, the characteristic $T(R,f)$ is increasing (Cartan's formula).
- If f is rational with $f = P/Q$, $\deg P = p$, $\deg Q = q$, then

$$T(R,f) = \max\{p,q\} \log R + O(1) \quad \text{as} \quad R \to \infty.$$

- In the converse direction,

$$\liminf_{R\to\infty} \frac{T(R,f)}{\log R} < \infty \quad \Rightarrow \quad f \text{ is rational.}$$

3. Rates of growth

- For all non-constant meromorphic f, the characteristic $T(R,f)$ is increasing (Cartan's formula).
- If f is rational with $f = P/Q$, $\deg P = p$, $\deg Q = q$, then

$$T(R,f) = \max\{p,q\} \log R + O(1) \quad \text{as} \quad R \to \infty.$$

- In the converse direction,

$$\liminf_{R\to\infty} \frac{T(R,f)}{\log R} < \infty \quad \Rightarrow \quad f \text{ is rational.}$$

- Many properties and results depend on the *order of growth*

$$\rho(f) = \limsup_{R\to\infty} \frac{\log T(R,f)}{\log R}$$

(the *lower order* $\lambda(f)$ has lim inf).

3. Rates of growth

- For all non-constant meromorphic f, the characteristic $T(R, f)$ is increasing (Cartan's formula).
- If f is rational with $f = P/Q$, $\deg P = p$, $\deg Q = q$, then

$$T(R, f) = \max\{p, q\} \log R + O(1) \quad \text{as} \quad R \to \infty.$$

- In the converse direction,

$$\liminf_{R\to\infty} \frac{T(R, f)}{\log R} < \infty \quad \Rightarrow \quad f \text{ is rational.}$$

- Many properties and results depend on the *order of growth*

$$\rho(f) = \limsup_{R\to\infty} \frac{\log T(R, f)}{\log R}$$

(the *lower order* $\lambda(f)$ has lim inf).

- For example,

$$\rho(e^z) = 1, \quad \rho(\tan(z^n)) = n, \quad \rho(\tan(e^z)) = \infty.$$

4. The first fundamental theorem

- If $a \in \mathbb{C}$ then $\log^+ |f - a| = \log^+ |f| + O(1)$ and, as $R \to \infty$,
$$m(R, f - a) = m(R, f) + O(1), \quad N(R, f - a) = N(R, f),$$

- If $a \in \mathbb{C}$ then $\log^+ |f - a| = \log^+ |f| + O(1)$ and, as $R \to \infty$,

$$m(R, f - a) = m(R, f) + O(1), \quad N(R, f - a) = N(R, f),$$

- and so, by Jensen's formula,

$$T(R, f) + O(1) = T(R, f - a) = T\left(R, \frac{1}{f - a}\right) + O(1).$$

4. The first fundamental theorem

- If $a \in \mathbb{C}$ then $\log^+ |f - a| = \log^+ |f| + O(1)$ and, as $R \to \infty$,

$$m(R, f - a) = m(R, f) + O(1), \quad N(R, f - a) = N(R, f),$$

- and so, by Jensen's formula,

$$T(R, f) + O(1) = T(R, f - a) = T\left(R, \frac{1}{f - a}\right) + O(1).$$

- This gives the *first fundamental theorem*

$$T(R, f) \quad = \quad m(R, a) + N(R, a) + O(1),$$

where

$$m(R, a) \quad = \quad m(R, a, f) = m\left(R, \frac{1}{f - a}\right),$$

$$N(R, a) \quad = \quad N(R, a, f) = N\left(R, \frac{1}{f - a}\right).$$

4. The first fundamental theorem

- Thus the total "affinity" to all values is the same insofar as: either f has a lot of a-points ($N(R, a)$ is large); or f is close to a on every circle ($m(R, a)$ is large).

4. The first fundamental theorem

- Thus the total "affinity" to all values is the same insofar as: either f has a lot of a-points ($N(R,a)$ is large); or f is close to a on every circle ($m(R,a)$ is large).
- For example, $f(z) = e^{2z} - e^z$ has

$$
\begin{aligned}
N(R,\infty) &= N(R,f) = 0, \\
m(R,\infty) &= m(R,f) \sim m(R,e^{2z}) = \frac{2R}{\pi}, \\
N(R,0) &= N(R,0,e^z - 1) = \frac{R}{\pi} + O(1), \\
m(R,0) &\sim m(R,e^{-z}) = \frac{R}{\pi}, \\
m(R,a) &= O(1), \quad N(R,a) = \frac{2R}{\pi} + O(1), \quad (a \in \mathbb{C} \setminus \{0\}).
\end{aligned}
$$

- An application (J. Clunie):
  Let f be transcendental meromorphic and let g be entire. Then
  $$T(r,g) = o(T(r, f \circ g)) \quad \text{as} \quad r \to \infty.$$

- An application (J. Clunie):
  Let f be transcendental meromorphic and let g be entire. Then
  $$T(r, g) = o(T(r, f \circ g)) \quad \text{as} \quad r \to \infty.$$
- *Proof.* Choose $a \in \mathbb{C}$ such that a is not a critical value of $h = f \circ g$ and f has infinitely many a-points $w_1, w_2, \ldots$.

- An application (J. Clunie):
  Let f be transcendental meromorphic and let g be entire. Then
  $$T(r,g) = o(T(r, f \circ g)) \quad \text{as} \quad r \to \infty.$$
- *Proof.* Choose $a \in \mathbb{C}$ such that a is not a critical value of $h = f \circ g$ and f has infinitely many a-points $w_1, w_2, \ldots$.
- Fix $N \in \mathbb{N}$ and choose $C, \delta > 0$ such that
  $$|w - w_j| < \delta \quad \Rightarrow \quad |f(w) - a| \leq C|w - w_j| \quad (j = 1, \ldots, N).$$

4. The first fundamental theorem

- An application (J. Clunie):
  Let f be transcendental meromorphic and let g be entire. Then
  $$T(r,g) = o(T(r,f\circ g)) \quad \text{as} \quad r \to \infty.$$
- *Proof.* Choose $a \in \mathbb{C}$ such that a is not a critical value of $h = f \circ g$ and f has infinitely many a-points $w_1, w_2, \ldots$.
- Fix $N \in \mathbb{N}$ and choose $C, \delta > 0$ such that
  $$|w - w_j| < \delta \quad \Rightarrow \quad |f(w) - a| \leq C|w - w_j| \quad (j = 1, \ldots, N).$$
- This gives
  $$\sum_{j=1}^{N} m(r, w_j, g) \leq m(r, a, h) + O(1).$$

5. The first fundamental theorem

- Also
$$\sum_{j=1}^{N} N(r, w_j, g) \leq N(r, a, h).$$

- Also
$$\sum_{j=1}^{N} N(r, w_j, g) \leq N(r, a, h).$$
- Adding and applying the first fundamental theorem yields
$$\sum_{j=1}^{N} T(r, w_j, g) \leq T(r, a, h) + O(1), \quad NT(r, g) \leq T(r, h) + O(1).$$

- Also
$$\sum_{j=1}^{N} N(r, w_j, g) \leq N(r, a, h).$$
- Adding and applying the first fundamental theorem yields
$$\sum_{j=1}^{N} T(r, w_j, g) \leq T(r, a, h) + O(1), \quad NT(r, g) \leq T(r, h) + O(1).$$
- Hence $T(r, g) = o(T(r, f \circ g))$.

- Also
$$\sum_{j=1}^{N} N(r, w_j, g) \leq N(r, a, h).$$
- Adding and applying the first fundamental theorem yields
$$\sum_{j=1}^{N} T(r, w_j, g) \leq T(r, a, h) + O(1), \quad NT(r, g) \leq T(r, h) + O(1).$$
- Hence $T(r, g) = o(T(r, f \circ g))$.
- **Corollary:** *Let f be a transcendental entire function and let $1 \leq m < n$. Then the iterates satisfy $T(r, f^m) = o(T(r, f^n))$ as $r \to \infty$.*

5. The second fundamental theorem

- The second fundamental theorem shows that, in the formula

$$T(R,f) = m(R,a) + N(R,a) + O(1),$$

the key term is $N(R,a)$ for most a.

5. The second fundamental theorem

- The second fundamental theorem shows that, in the formula

$$T(R,f) = m(R,a) + N(R,a) + O(1),$$

  the key term is $N(R,a)$ for most a.
- The 2FT depends on the *lemma of the logarithmic derivative*.

5. The second fundamental theorem

- The second fundamental theorem shows that, in the formula

$$T(R,f) = m(R,a) + N(R,a) + O(1),$$

the key term is $N(R,a)$ for most a.

- The 2FT depends on the *lemma of the logarithmic derivative*.
- Writing, as before,

$$g(w) = f(w) \prod_{j=1}^{m} \left(\frac{R(w-a_j)}{R^2 - \overline{a_j} w} \right)^{-1} \prod_{k=1}^{n} \left(\frac{R(w-b_k)}{R^2 - \overline{b_k} w} \right),$$

then Poisson's formula from §2 shows that

$$\log g(z) - \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\phi})| \left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right) d\phi$$

has zero real part for $|z| < R$ and so is constant there.

5. The second fundamental theorem

- The second fundamental theorem shows that, in the formula

$$T(R,f) = m(R,a) + N(R,a) + O(1),$$

  the key term is $N(R,a)$ for most a.
- The 2FT depends on the *lemma of the logarithmic derivative*.
- Writing, as before,

$$g(w) = f(w) \prod_{j=1}^{m} \left(\frac{R(w-a_j)}{R^2 - \overline{a_j} w} \right)^{-1} \prod_{k=1}^{n} \left(\frac{R(w-b_k)}{R^2 - \overline{b_k} w} \right),$$

  then Poisson's formula from §2 shows that

$$\log g(z) - \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\phi})| \left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right) d\phi$$

  has zero real part for $|z| < R$ and so is constant there.
- Differentiating gives a formula for g'/g and so for f'/f.

5. The second fundamental theorem

- The *differentiated Poisson-Jensen formula* reads

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi}-z)^2}\, d\phi \\ &+ \sum_{j=1}^{m}\left(\frac{\overline{a_j}}{R^2-\overline{a_j}z}+\frac{1}{z-a_j}\right) \\ &- \sum_{k=1}^{n}\left(\frac{\overline{b_k}}{R^2-\overline{b_k}z}+\frac{1}{z-b_k}\right)\end{aligned}$$

for $|z| < R$, $z \notin \{a_j, b_k\}$ (the zeros/poles in $|w| < R$).

5. The second fundamental theorem

- The *differentiated Poisson-Jensen formula* reads

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi}-z)^2}\, d\phi \\ &+ \sum_{j=1}^{m}\left(\frac{\overline{a_j}}{R^2-\overline{a_j}z}+\frac{1}{z-a_j}\right) \\ &- \sum_{k=1}^{n}\left(\frac{\overline{b_k}}{R^2-\overline{b_k}z}+\frac{1}{z-b_k}\right) \end{aligned}$$

for $|z|<R$, $z \notin \{a_j, b_k\}$ (the zeros/poles in $|w|<R$).

- This leads, for $|z| = r < R$, to

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2R(m(R,f)+m(R,1/f))}{(R-r)^2} + 2\sum \frac{1}{|z-A|} \quad ,$$

with the sum over all zeros and poles A of f in $|w|<R$.

5. The second fundamental theorem

- Provided z is not too close to a zero/pole,

$$\sum \frac{1}{|z-A|}$$

is not too much bigger than

$$n(R,f) + n(R,1/f) \leq \frac{N(S,f) + N(S,1/f)}{\log S/R} \quad (S > R),$$

5. The second fundamental theorem

- Provided z is not too close to a zero/pole,

$$\sum \frac{1}{|z-A|}$$

is not too much bigger than

$$n(R,f)+n(R,1/f) \leq \frac{N(S,f)+N(S,1/f)}{\log S/R} \quad (S>R),$$

- and $f'(z)/f(z)$ is not too much larger than $T(S,f)$.

5. The second fundamental theorem

- Provided z is not too close to a zero/pole,

$$\sum \frac{1}{|z-A|}$$

is not too much bigger than

$$n(R,f)+n(R,1/f) \leq \frac{N(S,f)+N(S,1/f)}{\log S/R} \quad (S>R),$$

- and $f'(z)/f(z)$ is not too much larger than $T(S,f)$.
- Growth lemmas yield the *lemma of the logarithmic derivative*:

$$m(r,f'/f) \leq C_1 \log^+ T(r,f) + C_2 \log r \quad (n.e.)$$

(*nearly everywhere*) i.e. *as $r \to \infty$ outside a set of finite measure.*

5. The second fundamental theorem

- Provided z is not too close to a zero/pole,

$$\sum \frac{1}{|z-A|}$$

is not too much bigger than

$$n(R,f)+n(R,1/f) \leq \frac{N(S,f)+N(S,1/f)}{\log S/R} \quad (S>R),$$

- and $f'(z)/f(z)$ is not too much larger than $T(S,f)$.
- Growth lemmas yield the *lemma of the logarithmic derivative*:

$$m(r,f'/f) \leq C_1 \log^+ T(r,f) + C_2 \log r \quad (n.e.)$$

(*nearly everywhere*) i.e. *as $r \to \infty$ outside a set of finite measure.*

- Sharp bounds investigated by Hinkkanen and others.

5. The second fundamental theorem

- Let $a_1, \ldots, a_q \in \mathbb{C}$ (distinct). Let

$$H(z) = \sum_{j=1}^{q} \frac{1}{f(z) - a_j}.$$

5. The second fundamental theorem

- Let $a_1, \ldots, a_q \in \mathbb{C}$ (distinct). Let

$$H(z) = \sum_{j=1}^{q} \frac{1}{f(z) - a_j}.$$

- Then the lemma of the logarithmic derivative gives

$$m(r, f'H) \leq \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + O(1) = S(r, f),$$

where

$$S(r, f) \quad \text{means} \quad O(\log^+ T(r, f) + \log r) \quad (n.e.).$$

5. The second fundamental theorem

- Let $a_1, \ldots, a_q \in \mathbb{C}$ (distinct). Let

$$H(z) = \sum_{j=1}^{q} \frac{1}{f(z) - a_j}.$$

- Then the lemma of the logarithmic derivative gives

$$m(r, f'H) \leq \sum_{j=1}^{q} m\left(r, \frac{f'}{f - a_j}\right) + O(1) = S(r, f),$$

where

$$S(r, f) \quad \text{means} \quad O(\log^+ T(r, f) + \log r) \quad (n.e.).$$

- On the other hand, for each j,

$$\log^+ \frac{1}{|f(z) - a_j|} \leq \log^+ |H(z)| + O(1).$$

5. The second fundamental theorem

- Take $\varepsilon > 0$ so that $E_j = \{z \in \mathbb{C} : |f(z) - a_j| < \varepsilon\}$ are disjoint.

- Take $\varepsilon > 0$ so that $E_j = \{z \in \mathbb{C} : |f(z) - a_j| < \varepsilon\}$ are disjoint.
- Then

$$\begin{aligned}
\sum_{j=1}^{q} m(r, a_j) &\leq \sum_{j=1}^{q} \left[\frac{1}{2\pi} \int_{[0,2\pi]\cap E_j} \log^+ \frac{1}{|f(re^{i\phi}) - a_j|} \, d\phi + \log \frac{1}{\varepsilon} \right] \\
&\leq \sum_{j=1}^{q} \frac{1}{2\pi} \int_{[0,2\pi]\cap E_j} \log^+ |H(re^{i\phi})| \, d\phi + O(1) \\
&\leq m(r, H) + O(1) = m(r, f'H/f') + O(1) \\
&\leq m(r, f'H) + m(r, 1/f') + O(1).
\end{aligned}$$

5. The second fundamental theorem

- Take $\varepsilon > 0$ so that $E_j = \{z \in \mathbb{C} : |f(z) - a_j| < \varepsilon\}$ are disjoint.
- Then

$$\begin{aligned}
\sum_{j=1}^{q} m(r, a_j) &\leq \sum_{j=1}^{q} \left[\frac{1}{2\pi} \int_{[0,2\pi]\cap E_j} \log^+ \frac{1}{|f(re^{i\phi}) - a_j|} \, d\phi + \log \frac{1}{\varepsilon} \right] \\
&\leq \sum_{j=1}^{q} \frac{1}{2\pi} \int_{[0,2\pi]\cap E_j} \log^+ |H(re^{i\phi})| \, d\phi + O(1) \\
&\leq m(r, H) + O(1) = m(r, f'H/f') + O(1) \\
&\leq m(r, f'H) + m(r, 1/f') + O(1).
\end{aligned}$$

- In particular we get the *key inequality*

$$\sum_{j=1}^{q} m(r, a_j) \leq m(r, 1/f') + S(r, f).$$

5. The second fundamental theorem

- Here

$$\begin{aligned} m(r,1/f') &= T(r,1/f') - N(r,1/f') \\ &= T(r,f') - N(r,1/f') + O(1) \quad \text{(Jensen)} \\ &= m(r,f') + N(r,f') - N(r,1/f') + O(1). \end{aligned}$$

5. The second fundamental theorem

- Here

$$\begin{aligned} m(r,1/f') &= T(r,1/f') - N(r,1/f') \\ &= T(r,f') - N(r,1/f') + O(1) \quad \text{(Jensen)} \\ &= m(r,f') + N(r,f') - N(r,1/f') + O(1). \end{aligned}$$

- But

$$\begin{aligned} m(r,f') &= m(r, f \cdot f'/f) \leq m(r,f) + m(r,f'/f) \\ &\leq m(r,f) + S(r,f). \end{aligned}$$

5. The second fundamental theorem

- Here

$$\begin{aligned} m(r,1/f') &= T(r,1/f') - N(r,1/f') \\ &= T(r,f') - N(r,1/f') + O(1) \quad \text{(Jensen)} \\ &= m(r,f') + N(r,f') - N(r,1/f') + O(1). \end{aligned}$$

- But

$$\begin{aligned} m(r,f') &= m(r,f\cdot f'/f) \leq m(r,f) + m(r,f'/f) \\ &\leq m(r,f) + S(r,f). \end{aligned}$$

- Now let $\overline{N}(r,f)$ count each pole exactly once: then

$$\begin{aligned} N(r,f') &= N(r,f) + \overline{N}(r,f) \\ &= 2N(r,f) - [N(r,f) - \overline{N}(r,f)]. \end{aligned}$$

- This gives the *second fundamental theorem* in the form

$$m(r,f) + \sum_{j=1}^{q} m(r,a_j) \leq 2T(r,f) - N_1(r) + S(r,f),$$

in which the *branching term*

$$N_1(r) = N(r,f) - \overline{N}(r,f) + N(r,1/f') \geq 0$$

counts multiple points of f.

- The *Nevanlinna deficiency* is

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

(using the first fundamental theorem).

5. The second fundamental theorem

- The *Nevanlinna deficiency* is

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, a)}{T(r, f)}$$

(using the first fundamental theorem).

- From the second fundamental theorem

$$m(r, f) + \sum_{j=1}^{q} m(r, a_j) \leq 2T(r, f) + S(r, f),$$

we get the deficiency relation

$$\sum_{(a)} \delta(a) \leq 2$$

(and hence Picard's theorem, since omitted values have $\delta(a) = 1$).

5. The second fundamental theorem

- **Example 1:** we have $\delta(0, e^{2z} - e^z) = 1/2$.

- **Example 1:** we have $\delta(0, e^{2z} - e^z) = 1/2$.
- **Example 2:** the error function satisfies

$$f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = 1 + O(|e^{-z^2}|) \quad \text{for} \quad |\arg z| \leq \pi/4,$$

$$\delta(\infty, f) = 1, \quad \delta(\pm 1, f) = \frac{1}{2}, \quad \sum_a \delta(a, f) = 2, \quad \rho(f) = 2.$$

5. The second fundamental theorem

- **Example 1:** we have $\delta(0, e^{2z} - e^z) = 1/2$.
- **Example 2:** the error function satisfies

$$f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = 1 + O(|e^{-z^2}|) \quad \text{for} \quad |\arg z| \leq \pi/4,$$

$$\delta(\infty, f) = 1, \quad \delta(\pm 1, f) = \frac{1}{2}, \quad \sum_a \delta(a, f) = 2, \quad \rho(f) = 2.$$

- **Example 3:** if P is a polynomial and

$$F = \frac{f_1}{f_2}, \quad f_j'' + Pf_j = 0, \quad W(f_1, f_2) \neq 0,$$

then $\sum \delta(a, F) = 2$, $\rho(F) = (\deg P + 2)/2$ and $N_1(r, F) = 0$.

- Drasin (ca. 1986) proved F. Nevanlinna's conjecture (1920s).
  If $\lambda(f) < \infty$ *and* $\sum \delta(a) = 2$ *then:*
  (i) $2\lambda(f) = 2\rho(f)$ *is in* $\{2, 3, 4, \ldots\}$*;*
  (ii) $\rho(f)\delta(a) \in \mathbb{Z}$ *for each* a*;*
  (iii) if $\delta(a) > 0$ *then* $f(z)$ *is close to* a *in a "sector".*

5. The second fundamental theorem

- Drasin (ca. 1986) proved F. Nevanlinna's conjecture (1920s).
  If $\lambda(f) < \infty$ *and* $\sum \delta(a) = 2$ *then:*
  (i) $2\lambda(f) = 2\rho(f)$ *is in* $\{2, 3, 4, \ldots\}$*;*
  (ii) $\rho(f)\delta(a) \in \mathbb{Z}$ *for each* a*;*
  (iii) if $\delta(a) > 0$ *then* $f(z)$ *is close to* a *in a "sector".*
- Combining the 2FT and a result of Eremenko (ca. 1991):
  if $\lambda(f) < \infty$ *then the following are equivalent:*
  (a) $\sum \delta(a) = 2$ (maximal deficiency sum)
  (b) $N_1(r) = o(T(r))$ (i.e. f has few multiple points).

5. The second fundamental theorem

- For an alternative form of the second fundamental theorem add $N(r,f) + \sum_{j=1}^{q} N(r,a_j)$ to both sides of

$$m(r,f) + \sum_{j=1}^{q} m(r,a_j) \leq 2T(r,f) - N_1(r) + S(r,f).$$

5. The second fundamental theorem

- For an alternative form of the second fundamental theorem add $N(r,f) + \sum_{j=1}^{q} N(r,a_j)$ to both sides of

$$m(r,f) + \sum_{j=1}^{q} m(r,a_j) \le 2T(r,f) - N_1(r) + S(r,f).$$

- Since $m(r,a_j) + N(r,a_j) = T(r,f) + O(1)$ by the 1FT,

$$\begin{aligned}(q-1)T(r,f) &\le N(r,f) + \sum_{j=1}^{q} N(r,a_j) - N_1(r) + S(r,f) \\ &\le \overline{N}(r,f) + \sum_{j=1}^{q} \overline{N}(r,a_j) + S(r,f).\end{aligned}$$

5. The second fundamental theorem

- For an alternative form of the second fundamental theorem add $N(r,f)+\sum_{j=1}^{q} N(r,a_j)$ to both sides of

$$m(r,f)+\sum_{j=1}^{q} m(r,a_j) \le 2T(r,f) - N_1(r) + S(r,f).$$

- Since $m(r,a_j)+N(r,a_j) = T(r,f)+O(1)$ by the 1FT,

$$\begin{aligned}(q-1)T(r,f) &\le N(r,f)+\sum_{j=1}^{q} N(r,a_j) - N_1(r) + S(r,f)\\ &\le \overline{N}(r,f)+\sum_{j=1}^{q} \overline{N}(r,a_j) + S(r,f).\end{aligned}$$

- This gives the deficiency relation with $\overline{N}(r,a)$:

$$\sum_{(a)} \Theta(a) \le 2, \quad \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)}.$$

6. A method of Teichmüller I

This relates value distribution to critical and asymptotic values.

- Let D be a simply connected domain in $\mathbb{C} \setminus \{0\}$ and let G map D conformally onto $B(0, 1)$, with $G(z_0) = 0$, $z_0 \in D$.

This relates value distribution to critical and asymptotic values.

- Let D be a simply connected domain in $\mathbb{C} \setminus \{0\}$ and let G map D conformally onto $B(0, 1)$, with $G(z_0) = 0$, $z_0 \in D$.
- Let the arc Ω given by $z = re^{i\theta}$, $\theta_1 < \theta < \theta_2$, lie in D, with its endpoints $re^{i\theta_j} \in \partial D$, $j = 1, 2$.

6. A method of Teichmüller I

This relates value distribution to critical and asymptotic values.

- Let D be a simply connected domain in $\mathbb{C} \setminus \{0\}$ and let G map D conformally onto $B(0,1)$, with $G(z_0) = 0$, $z_0 \in D$.
- Let the arc Ω given by $z = re^{i\theta}$, $\theta_1 < \theta < \theta_2$, lie in D, with its endpoints $re^{i\theta_j} \in \partial D$, $j = 1, 2$.
- Teichmüller estimates

$$\int_{\theta_1}^{\theta_2} \log^+ \frac{1}{|G(re^{i\theta})|} \, d\theta.$$

This relates value distribution to critical and asymptotic values.

- Let D be a simply connected domain in $\mathbb{C} \setminus \{0\}$ and let G map D conformally onto $B(0,1)$, with $G(z_0) = 0$, $z_0 \in D$.
- Let the arc Ω given by $z = re^{i\theta}$, $\theta_1 < \theta < \theta_2$, lie in D, with its endpoints $re^{i\theta_j} \in \partial D$, $j = 1, 2$.
- Teichmüller estimates

$$\int_{\theta_1}^{\theta_2} \log^+ \frac{1}{|G(re^{i\theta})|} \, d\theta.$$

- First seek an upper bound for

$$\log^+ \frac{1}{|w|}, \quad w = G(z), \quad z \in \Omega = \{re^{i\theta} : \quad \theta_1 < \theta < \theta_2\}.$$

WLOG $|w| < 1/2$.

6. A method of Teichmüller I

- The image of $B(w, 1-|w|)$ under $\zeta = H(w) = \log G^{-1}(w)$ contains neither of the points $\log r + i\theta_j$,
  so Koebe's 1/4 theorem gives

$$|H'(w)| \leq C \min\{\theta - \theta_j\} \leq C(\theta_2 - \theta_1).$$

6. A method of Teichmüller I

- The image of $B(w, 1-|w|)$ under $\zeta = H(w) = \log G^{-1}(w)$ contains neither of the points $\log r + i\theta_j$,
  so Koebe's 1/4 theorem gives

$$|H'(w)| \leq C \min\{\theta - \theta_j\} \leq C(\theta_2 - \theta_1).$$

- By Koebe's distortion theorem,

$$|H'(u)| \leq C(\theta_2 - \theta_1) \quad \text{for} \quad |u| \leq \frac{1}{2}.$$

6. A method of Teichmüller I

- The image of $B(w, 1-|w|)$ under $\zeta = H(w) = \log G^{-1}(w)$ contains neither of the points $\log r + i\theta_j$,
  so Koebe's 1/4 theorem gives

$$|H'(w)| \leq C \min\{\theta - \theta_j\} \leq C(\theta_2 - \theta_1).$$

- By Koebe's distortion theorem,

$$|H'(u)| \leq C(\theta_2 - \theta_1) \quad \text{for} \quad |u| \leq \frac{1}{2}.$$

- Hence, with $\theta_0 = \mathrm{Im}\,(\log z_0)$,

$$|\theta - \theta_0| \leq \left|\log \frac{z}{z_0}\right| = |H(w) - H(0)| \leq C|w|(\theta_2 - \theta_1).$$

6. A method of Teichmüller I

- The image of $B(w, 1-|w|)$ under $\zeta = H(w) = \log G^{-1}(w)$ contains neither of the points $\log r + i\theta_j$,
  so Koebe's $1/4$ theorem gives

$$|H'(w)| \leq C \min\{\theta - \theta_j\} \leq C(\theta_2 - \theta_1).$$

- By Koebe's distortion theorem,

$$|H'(u)| \leq C(\theta_2 - \theta_1) \quad \text{for} \quad |u| \leq \frac{1}{2}.$$

- Hence, with $\theta_0 = \mathrm{Im}\,(\log z_0)$,

$$|\theta - \theta_0| \leq \left|\log \frac{z}{z_0}\right| = |H(w) - H(0)| \leq C|w|(\theta_2 - \theta_1).$$

- This gives

$$\log^+ \frac{1}{|G(z)|} \leq C + \log^+ \left|\frac{\theta_2 - \theta_1}{\theta - \theta_0}\right|.$$

6. A method of Teichmüller I

- But

$$\begin{aligned} I &:= \int_{\theta_1}^{\theta_2} \log^+ \left| \frac{\theta_2 - \theta_1}{\theta - \theta_0} \right| d\theta \\ &= (\theta_2 - \theta_1) \int_0^1 \log^+ \frac{1}{|\mu + x|} \, dx \quad \left(\mu = \frac{\theta_1 - \theta_0}{\theta_2 - \theta_1} \right) \\ &\leq (\theta_2 - \theta_1) \int_{\mathbb{R}} \log^+ \frac{1}{|y|} \, dy = C(\theta_2 - \theta_1). \end{aligned}$$

- But

$$\begin{aligned} I &:= \int_{\theta_1}^{\theta_2} \log^+ \left| \frac{\theta_2 - \theta_1}{\theta - \theta_0} \right| \, d\theta \\ &= (\theta_2 - \theta_1) \int_0^1 \log^+ \frac{1}{|\mu + x|} \, dx \quad \left(\mu = \frac{\theta_1 - \theta_0}{\theta_2 - \theta_1} \right) \\ &\leq (\theta_2 - \theta_1) \int_{\mathbb{R}} \log^+ \frac{1}{|y|} \, dy = C(\theta_2 - \theta_1). \end{aligned}$$

- This gives Teichmüller's estimate

$$\int_{\theta_1}^{\theta_2} \log^+ \frac{1}{|G(re^{i\theta})|} \, d\theta \leq C(\theta_2 - \theta_1).$$

6. A method of Teichmüller I

- But

$$\begin{aligned} I &:= \int_{\theta_1}^{\theta_2} \log^+ \left| \frac{\theta_2 - \theta_1}{\theta - \theta_0} \right| \, d\theta \\ &= (\theta_2 - \theta_1) \int_0^1 \log^+ \frac{1}{|\mu + x|} \, dx \quad \left(\mu = \frac{\theta_1 - \theta_0}{\theta_2 - \theta_1} \right) \\ &\leq (\theta_2 - \theta_1) \int_{\mathbb{R}} \log^+ \frac{1}{|y|} \, dy = C(\theta_2 - \theta_1). \end{aligned}$$

- This gives Teichmüller's estimate

$$\int_{\theta_1}^{\theta_2} \log^+ \frac{1}{|G(re^{i\theta})|} \, d\theta \leq C(\theta_2 - \theta_1).$$

- The same holds with G replaced by G^m for $m \in \mathbb{N}$, $m \leq m_0$ (fixed).

6. A method of Teichmüller I

- **Collingwood's theorem (as refined by Teichmüller)**
  Suppose f and a are such that:
  the spherical disc $\chi(w, a) < \delta$ contains no asymptotic values;
  the punctured disc $0 < \chi(w, a) < \delta$ contains no critical values;
  all a-points have multiplicity at most $m_0 < \infty$.
  Then

$$m(r, a) = m(r, a, f) = O(1) \quad \text{as} \quad r \to \infty.$$

  In particular, $\delta(a, f) = 0$.

6. A method of Teichmüller I

- **Collingwood's theorem (as refined by Teichmüller)**
  Suppose f and a are such that:
  the spherical disc $\chi(w,a) < \delta$ contains no asymptotic values;
  the punctured disc $0 < \chi(w,a) < \delta$ contains no critical values;
  all a-points have multiplicity at most $m_0 < \infty$.
  Then

$$m(r,a) = m(r,a,f) = O(1) \quad \text{as} \quad r \to \infty.$$

  In particular, $\delta(a,f) = 0$.
- *Proof.* Assume WLOG that the disc is $B(0,2)$.
  On each component D_k of $f^{-1}(B(0,1))$ write $f = G^m$
  with $G : D_k \to B(0,1)$ conformal and $m = m(k) \leq m_0$.
  Summing over those k for which D_k meets $|z| = r$,
  Teichmüller's estimate gives

$$m(r,1/f) \leq C \sum_k (\theta_2(k) - \theta_1(k)) \leq C2\pi.$$

7. A method of Teichmüller II: the derivative

- Let f be transcendental, in the Speiser class S, i.e. all finite asymptotic and critical values lie in $\{a_1, \ldots, a_q\}$.
  Let $\delta > 0$ be small.

- Let f be transcendental, in the Speiser class S, i.e. all finite asymptotic and critical values lie in $\{a_1, \ldots, a_q\}$.
  Let $\delta > 0$ be small.
- First, a well known inequality gives $R > 0$ such that

$$\left| \frac{zf'(z)}{f(z)} \right| \geq C \log^+ \frac{|f(z)|}{R} \quad \text{for } z \text{ large}$$

(take a component of $\{z \in \mathbb{C} : |f(z)| > R\}$ and apply Bloch's theorem to $\log f^{-1}(e^w)$ on $\mathrm{Re}\, w > \log R$).

7. A method of Teichmüller II: the derivative

- Let f be transcendental, in the Speiser class S, i.e. all finite asymptotic and critical values lie in $\{a_1, \ldots, a_q\}$.
  Let $\delta > 0$ be small.
- First, a well known inequality gives $R > 0$ such that

$$\left| \frac{zf'(z)}{f(z)} \right| \geq C \log^+ \frac{|f(z)|}{R} \quad \text{for } z \text{ large}$$

  (take a component of $\{z \in \mathbb{C} : |f(z)| > R\}$ and apply Bloch's theorem to $\log f^{-1}(e^w)$ on $\mathrm{Re}\, w > \log R$).
- Thus, if z is large and $|f(z)| > 1/\delta$,

$$\left| \frac{zf'(z)}{f(z)} \right| \geq 1, \quad \log \frac{1}{|f'(z)|} \leq \log |z|. \tag{1}$$

7. A method of Teichmüller II: the derivative

- Second, the same inequality applied to $1/(f - a_j)$ yields, for z large and $|f(z) - a_j| < \delta$,

$$\left| \frac{zf'(z)}{f(z) - a_j} \right| \geq 1, \quad \log \frac{1}{|f'(z)|} \leq \log |z| + \log \frac{1}{|f(z) - a_j|}. \quad (2)$$

- Second, the same inequality applied to $1/(f - a_j)$ yields, for z large and $|f(z) - a_j| < \delta$,

$$\left| \frac{zf'(z)}{f(z) - a_j} \right| \geq 1, \quad \log \frac{1}{|f'(z)|} \leq \log |z| + \log \frac{1}{|f(z) - a_j|}. \quad (2)$$

- Third, if the disc $B(a, \delta) \subseteq \mathbb{C}$ does not meet $\{a_j\}$ then

$$\log^+ \frac{1}{|f'(z)|} \leq \log |z| + C \quad (3)$$

for z large and $|f(z) - a| < \delta/2$.
(To see this, apply Koebe's $1/4$ theorem to $\log f^{-1}(v)$ on $B(a, 3\delta/4)$ as in the proof of Collingwood's theorem).

7. A method of Teichmüller II: the derivative

- Now we can cover the Riemann sphere by finitely many disjoint E_j such that

$$z \text{ large, } w = f(z) \in E_j \Rightarrow \text{one of (1), (2) and (3).}$$

7. A method of Teichmüller II: the derivative

- Now we can cover the Riemann sphere by finitely many disjoint E_j such that

$$z \text{ large}, \ w = f(z) \in E_j \Rightarrow \text{one of (1), (2) and (3).}$$

- This gives Teichmüller's second fundamental theorem for S: *if f is transcendental and all finite asymptotic and critical values lie in $\{a_1, \ldots, a_q\}$, then, as $r \to \infty$,*

$$m(r, 1/f') \leq \sum_{j=1}^{q} m(r, a_j) + O(\log r) \leq m(r, 1/f') + S(r, f).$$

7. A method of Teichmüller II: the derivative

- Now we can cover the Riemann sphere by finitely many disjoint E_j such that

$$z \text{ large, } w = f(z) \in E_j \Rightarrow \text{one of (1), (2) and (3).}$$

- This gives Teichmüller's second fundamental theorem for S: *if f is transcendental and all finite asymptotic and critical values lie in $\{a_1, \ldots, a_q\}$, then, as $r \to \infty$,*

$$m(r, 1/f') \leq \sum_{j=1}^{q} m(r, a_j) + O(\log r) \leq m(r, 1/f') + S(r, f).$$

- Also

$$(q-1)T(r, f) = \overline{N}(r, f) + \sum_{j=1}^{q} \overline{N}(r, a_j) + S(r, f)$$

(using $m(r, f') \leq m(r, f) + S(r, f) \leq m(r, f') + S(r, f)$).

8. Growth and singular values

- Critical and asymptotic values also influence *growth*: an entire function f in the class B (i.e. set of finite asymptotic and critical values is bounded) must have $\lambda(f) \geq 1/2$.

8. Growth and singular values

- Critical and asymptotic values also influence *growth*: an entire function f in the class B (i.e. set of finite asymptotic and critical values is bounded) must have $\lambda(f) \geq 1/2$.
- If not the $\cos \pi\lambda$ theorem gives circles $|z| = r_n \to \infty$ on which $f(z) \to \infty$, contradicting the fact that components of $\{z : |f(z)| > R \gg 1\}$ are simply connected.

8. Growth and singular values

- Critical and asymptotic values also influence *growth*: an entire function f in the class B (i.e. set of finite asymptotic and critical values is bounded) must have $\lambda(f) \geq 1/2$.
- If not the $\cos \pi\lambda$ theorem gives circles $|z| = r_n \to \infty$ on which $f(z) \to \infty$, contradicting the fact that components of $\{z : |f(z)| > R \gg 1\}$ are simply connected.
- A meromorphic function in B can grow arbitrarily slowly: take

$$\frac{1}{f(z)} = g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right),$$

constructed so that critical values of g tend to ∞.

8. Growth and singular values

- A meromorphic f in class S always has

$$L(f) := \liminf_{r \to \infty} \frac{T(r, f)}{(\log r)^2} > 0.$$

8. Growth and singular values

- A meromorphic f in class S always has

$$L(f) := \liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^2} > 0.$$

- Eremenko 2004: *if there are just three singular values then* $L(f) \geq \sqrt{3}/2\pi$.

9. Sources

W.K. Hayman, *Meromorphic functions*
A.A. Gol'dberg and I.V. Ostrovskii, *Value distribution of meromorphic functions*
R. Nevanlinna, *Eindeutige analytische Funktionen*
G. Jank and L. Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*