Connectedness properties of the set where the iterates of an entire function are bounded

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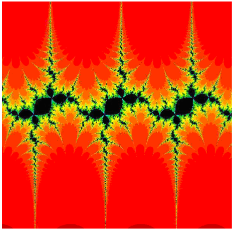
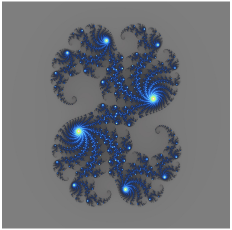
Basic properties of $K(f)$ for entire functions

NON-LINEAR POLYNOMIAL

- $J(f) = \partial K(f)$
- $K(f)$ compact (the filled Julia set)
- $K(f) = \mathbb{C} \setminus I(f)$





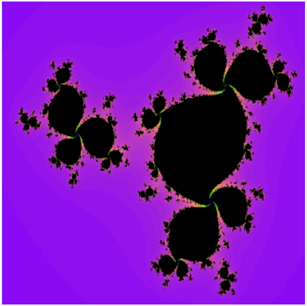
TRANSCENDENTAL ENTIRE

- $J(f) = \partial K(f)$
- $K(f)$ neither closed nor bounded
- $\mathbb{C} \setminus (I(f) \cup K(f))$ always contains points in $J(f)$ (and sometimes points in $F(f)$).

Connectedness of $K(f)$ for polynomials

Theorem

If f is a non-linear polynomial, then $K(f)$ is either connected or has uncountably many components.



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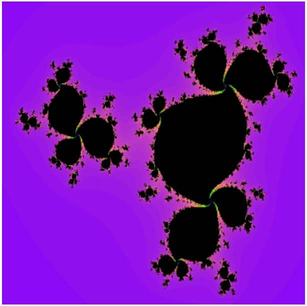

Theorem (Kozlovski and van Strien, Qiu and Yin 2009)

If f is a non-linear polynomial, a component of $K(f)$ is a singleton if and only if its orbit includes no periodic component of $K(f)$ containing a critical point.

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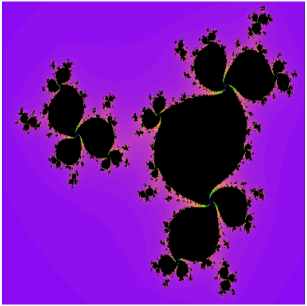

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If f is a non-linear polynomial, a component of $K(f)$ is a singleton if and only if its orbit includes no periodic component of $K(f)$ containing a critical point.

Corollary (Branner-Hubbard conjecture)

$K(f)$ is totally disconnected if and only if every component of $K(f)$ containing a critical point is aperiodic.

Strongly polynomial-like functions

A *polynomial-like* map $(f; V, W)$ of degree d is a mapping f from one topological disc V to another W, where $\overline{V} \subset W$, such that each point in W has exactly d preimages in V. Its *filled Julia set* is

$$K(f; V, W) = \bigcap_{k \geq 0} f^{-k}(V).$$

Strongly polynomial-like functions

Definition

We call a transcendental entire function f *strongly polynomial-like* if there exist sequences $(V_n), (W_n)$ of topological discs such that

$$V_n \subset V_{n+1} \text{ and } W_n \subset W_{n+1}, n \in \mathbb{N},$$

$$\bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} W_n = \mathbb{C},$$

and each of the triples $(f; V_n, W_n)$ is a polynomial-like map.

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Number of components of $K(f)$

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(c) *If* f *is strongly polynomial-like, then either* $K(f)$ *is connected or else every neighbourhood of a point in* $J(f)$ *meets uncountably many components of* $K(f)$.

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- Question: does (c) hold for a general transcendental entire function?

Nature of components of $K(f)$

Theorem B

Let f be a strongly polynomial-like transcendental entire function and let K be a component of $K(f)$.

(a) *The component K is a singleton if and only if the orbit of K includes no periodic component of $K(f)$ containing a critical point. In particular, if K is a wandering component of $K(f)$, then K is a singleton.*

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Corollary

Let f be a strongly polynomial-like transcendental entire function.

(a) *All but countably many components of $K(f)$ are singletons.*

(b) *$K(f)$ is totally disconnected if and only if each component of $K(f)$ containing a critical point is aperiodic.*

Many functions are strongly polynomial-like

Theorem C

A transcendental entire function f is strongly polynomial-like if there exists an unbounded sequence (r_n) of positive real numbers such that

$$m(r_n, f) := \min\{|f(z)| : |z| = r_n\} > r_n, \quad \textit{for } n \in \mathbb{N}.$$

In particular, this is the case if one of the following conditions holds:

(a) *f has a multiply connected Fatou component;*

(b) *f has growth not exceeding order $\frac{1}{2}$, minimal type;*

(c) *f has finite order and Fabry gaps;*

(d) *f has a sufficiently strong version of the pits effect.*

Simple examples

$K(f)$ is connected for

- $\lambda e^z, 0 < \lambda < 1/e$
- $\sin z$
- $\cos z + z$ (*)

$K(f)$ is totally disconnected for

- $z + 1 + e^{-z}$ (*)
- $e^z + 2z$ (*)

* strongly polynomial-like functions.

Theorem D

A transcendental entire function f is strongly polynomial-like if and only if there exists a sequence of bounded, simply connected domains $(D_n)_{n\in\mathbb{N}}$ such that

- $\overline{D_n} \subset D_{n+1}$, *for* $n \in \mathbb{N}$,
- $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{C}$, *and*
- $f(\partial D_n)$ *surrounds* $\overline{D_n}$, *for* $n \in \mathbb{N}$.