OPEN PROBLEMS SUBMITTED AT (OR FOLLOWING) THE WORKSHOP ON "THE ROLE OF COMPLEX ANALYSIS IN COMPLEX DYNAMICS"

1. Problems submitted by Alex Eremenko

1. Let $f : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$, where f is a branch of an algebraic function. Does f have finitely many attracting cycles on $\hat{\mathbb{R}}$?

2. Let p be a polynomial of degree m and q be a polynomial of degree n, where $n > m$. It is known that the number of solutions of

$$p(\overline{z}) = q(z)$$

is at most n^2 and conjectured that the number of solutions is at most $3n + c(m)$. This is known to be true if $m = 1$ but is open otherwise.

2. Problem submitted by Mitsu Shishikura

For a round annulus $A_R = \{z \in \mathbb{C} | 1 < |z| < R\}$, one can define its stretching deformation of factor (or dilatation) K to be a new conformal structure induced by a qc-mapping $h_K : A_R \to A_{R^K}$, where $h_K(z) = |z|^{K-1}z$. For an annulus in general, a conformal mapping to the round one will define the stretching deformation. Suppose an annulus A and a collection of disjoint subannuli $A_i \subset A$ ($i \in I$, where I is some index set, possibly countably many) are given. The stretching deformations of the subannuli A_i by factor K define a new conformal structure on A and use the standard conformal structure on $A \setminus \cup_{i\in I} A_i$. Let us define A^K to be A with this new conformal structure. By Grötzsch inequality, we have

$$mod A^K \geq K \sum_{i \in I_{ess}} mod A_i^K,$$

where I_{ess} is the collection of indices corresponding to essential (homotopically non-trivial) subannuli. The question is when K tends to ∞, whether one can show an opposite inequality

$$mod A^K \leq K \sum_{i \in I_{ess}} mod A_i^K + o(K),$$

possibly with a more accurate estimate on the small term (e.g. O(1)). This question is related to the limit of qc-deformations of rational maps (or entire functions etc.). Within (super)attracting basins, Siegel disks, Herman rings (but not parabolic basins), one can define dynamically defined annuli (e.g. in an attracting basin, the linearization coordinate, divide the plane by circles going through the grand critical orbits). So the “stretching” qc-deformation can be considered as just a stretching on disjoint annuli. A positive answer to the above question leads to an analysis of how a rational map degenerates to a lower degree map as a result of the stretching deformation.

3. Problems submitted by Phil Rippon and Gwyneth Stallard

Let f be a transcendental entire function and U be a Fatou component in the escaping set $I(f)$.

1. If U is a Baker domain, are there always points in $\partial U \cap I(f)$?

2. If U is a wandering domain in $I(f)$ then there are known to be many points in $\partial U \cap I(f)$ and all known examples have $\partial U \subset I(f)$. Is there an example of a bounded/ unbounded wandering domain in $I(f)$ with $\partial U \cap I(f)^c \neq \emptyset$?

4. Problem submitted by John Mayer

The residual Julia set consists of those points in the Julia set which do not lie on the boundary of a Fatou component. (Such a point is said to be a buried point.) If the residual Julia set is totally disconnected, does it have topological dimension zero? [1]

Alex Eremenko commented that, if f is rational and Makienko's conjecture fails, then the residual Julia set is an indecomposable continuum [2]. (Makienko's conjecture is that the residual Jula set is empty if and only if one of the Fatou components is completely invariant.)

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5. Problem submitted by Lasse Rempe-Gillen

Adam Epstein has defined a class of finite type maps which is conformally natural and generalises rational maps, class S etc. The definition is independent of any ambient surface and is invariant under pre-composition by a conformal isomorphism. The problem is to find a class that is conformally natural and generalises finite type maps and meromorphic functions.

6. Problem submitted by David Drasin

Quasiconformal mappings became associated with classical value-distribution (Nevanlinna) theory almost from the beginning; while the formal definition is due to Grötzsch, in 1935 Ahlfors directly used it in the final part of his work on covering surfaces, and a little earlier on his study of functions of deficiency sum two. Further progress on the 'inverse problem' of this theory depended on quasiconformal compositions, with Teichmüller introducing a unified method of attack. By pursuing this viewpoint, the complete solution to Nevanlinna's inverse problem for meromorphic functions in the plane was obtained by Drasin (1977). The relevance of quasiconformal mappings to this process is simple to see when one tries to modify the simple example $f(z) = f_k(z) = \exp(z^k)$ by modifying f in the k tracts on which $f \to 0$ so that the limiting deficient values $\{a_j\}_1^k$ can be preassigned.

More recently, it was shown that the stronger second fundamental theorem

$$\sum_a \delta(a,f) + \theta(a,f) \leq 2$$

holds, even when summing over all *functions* $a(z)$ which grow slowly in comparison to f [K. Yamanoi, MR 2096455]. However, producing examples in this more general setting seems far more difficult, as one can see even from attempts to modify f_k to now approach k preassigned polynomials in these tracts. Efforts to do this via improved versions of the Teichmüller-Wittich-Belinskii theorem thus far have been unsuccessful (some flexibility in this T-W-B theorem has been discussed in the presentation by X. Jarque at this conference).

I believe I once saw a manuscript with W. H. J. Fuchs as co-author in which this stronger form was obtained for f entire and the $\{a_j\}$ polynomials, using the direct construction discussed in Chapter 4 of W. Hayman's text 'Meromorphic Functions', but this is not in Mathscinet. This method had been used to solve the inverse problem $\sum_{|a|<\infty} \delta(a,f) \leq 1$ for entire functions (and is in several standard texts), but has not been able to handle the general case for general meromorphic functions.

7. Problems on quasiregular mappings submitted by Alastair Fletcher, Dan Nicks and David Drasin

1. For a quasiregular mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ of transcendental type, can the Julia set $J(f)$ be characterized as the boundary of the fast escaping set $\partial A(f)$? See [2] for the fast escaping set in the context of quasiregular mappings.

2. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping of transcendental type. Fix R large and, following Rippon and Stallard [5], define the annuli

$$A_n = \{x \in \mathbb{R}^m : M^{n-1}(R,f) \leq |x| < M^n(R,f)\}$$

for $n \in \mathbb{N}$, where $M^j(R,f)$ denotes the iterated maximum modulus. Given $x \in \mathbb{R}^m$, we say that x has the annular itinerary $(s_j)_{j=1}^{\infty}$ if and only if $f^j(x) \in A_{s_j}$. If $x \in A(f)$ then it has the itinerary $(s_0, s_0+1, s_0+2, \ldots)$. Which possible itineraries can arise?

3. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a quasiregular mapping which is either of transcendental type, or polynomial type with inner dilatation smaller than the degree. If $O^-(x)$ is an infinite set, is $\mathrm{cap}(\overline{O^-(x)}) > 0$? If this question has a positive answer then the Julia set has many properties associated with the Julia set of a holomorphic function. Note that the answer is positive in certain cases, see [1, 3] for more details.

4. Show that given an analytic set $E \subset \mathbb{R}^m$, that is, a continuous image of a Borel set, there exists a quasiregular mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ of transcendental type with the set of asymptotic values of f coinciding with E. Further, find a systematic way of constructing quasimeromorphic mappings $\mathbb{R}^m \to \overline{\mathbb{R}^m}$ which have arbitrarily slow growth and for which the set of asymptotic values coincides with E.

5. Produce a concrete example of the Julia set of a quasiregular mapping which is not uniformly quasiregular. Is there an example of a quasiregular mapping for which $\mathbb{R}^m \setminus J(f)$ contains a wandering domain? See [4] for an example in the plane where $\mathbb{R}^2 \setminus \partial I(f)$ contains

a wandering domain.

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8. Problems on the hyperbolic metric and wandering domains submitted by Jian-Hua Zheng

1. A hyperbolic domain U is called a BP domain after A. F. Beardon and Ch. Pommerenke if

$$C_U = \inf\{\lambda_U(z)\delta_U(z) : \ z \in U\} > 0,$$

where $\lambda_U(z)$ is the hyperbolic density at z of U and $\delta_U(z)$ is the Euclidean distance from z to ∂U; a universal covering mapping f from the unit disk $\mathbb{D}$ onto a BP domain is called a BP function on $\mathbb{D}$, that is to say, $C_{f(\mathbb{D})} > 0$.

Problem 1. For a BP function $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, should we have

$$|a_2| \leq \frac{1}{2C_{f(\mathbb{D})}}? \tag{8.1}$$

If (8.1) holds, then in terms of that C_U is invariant under the transformation $w = az + b$, we can establish the following distortion theorem for a BP function:

Let $f(z)$ be a BP function on $\mathbb{D}$. Then

$$|f'(0)|\frac{(1-|z|)^{1/d-1}}{(1+|z|)^{1/d+1}} \leq |f'(z)| \leq |f'(0)|\frac{(1+|z|)^{1/d-1}}{(1-|z|)^{1/d+1}}, \tag{8.2}$$

where $d = 2C_{f(\mathbb{D})} > 0$.

2. Let U be a wandering domain of a transcendental meromorphic function $f(z)$ such that $f^n|_U \to \infty (n \to \infty)$. Define after Bergweiler, Rippon and Stallard [7] for $z_0 \in U$ the functions

$$\overline{h}_U(z) = \limsup_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|}$$

and

$$\underline{h}_U(z) = \liminf_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|}, \ \ z \in U.$$

If $\overline{h}_U(z) = \underline{h}_U(z)$, we write the common value as $h_U(z)$.

Bergweiler, Rippon and Stallard [7] proved that if U is a multiply connected Fatou component of an entire function f, then $h_U(z)$ exists and is a positive non-constant harmonic function on U.

Problem 2. For a transcendental meromorphic function $f(z)$ with infinitely many poles, under what condition on the number of poles, for example, $T(r,f) \geq N(r,f)\log r$, does $h_U(z)$ exist?

In fact, if $\overline{h}_U(z) \not\equiv 1$ or $\underline{h}_U(z) \not\equiv 1$ and $T(r,f) \geq N(r,f)\log r$, then we can prove that for all sufficiently large n, $U_n = f^n(U) \supset \{z : r_n < |z| < r_n^c\}$ where $r_n \to \infty$ and $c > 1$.

3. In [11], we proved that if $f^n|_U \to \infty$ for a Fatou component U, then for any compact subset W of U, there exists a $M(W) > 1$ such that

$$M(W)^{-1}|f^n(z)| \leq |f^n(w)| \leq M(W)|f^n(z)|,\ \forall\ z, w \in W \tag{8.3}$$

provided that $\cup_{n=1}^{\infty} f^n(U)$ contains no sequence of round annuli D_m centered at 0 such that $\mathrm{dist}(0, D_m) \to \infty$ and $\mathrm{mod}(D_m) \to \infty$.

Problem 3. Does (8.3) hold for any compact subset W of a wandering domain U if $\cup_{n=1}^{\infty} f^n(U)$ contains a sequence of round annuli D_m centered at 0 such that $\mathrm{dist}(0, D_m) \to \infty$ and $\mathrm{mod}(D_m) \to \infty$.

(8.3) is always true for a Baker domain U, as proved by Baker [4] for the simply connected case, Zheng [12] and Rippon [10] for the general case. However, there exists an invariant Baker domain containing a sequence of round annuli D_m centered at 0 such that $\mathrm{dist}(0, D_m) \to \infty$ and $\mathrm{mod}(D_m) \to \infty$. Therefore, in Problem 3, we consider only wandering domains.

4. As we know, if the Julia set of a meromorphic function not of the form $\alpha+(z-\alpha)^{-k}e^{g(z)}$ is disconnected on the extended complex plane, then the Julia set has an uncountable number of components, and if the Fatou set has no completely invariant components, the Julia set has an uncountable number of buried components([9]). A Julia component J_0 is called wandering if $f^n(J_0) \cap f^m(J_0) = \emptyset$ for $n \neq m$, otherwise it is called periodic or preperiodic.

Problem 4. Does the Julia set of a meromorphic function have at most countably many periodic or preperiodic components? Is a wandering buried component of a Julia set a Jordan arc or a single point? Are all but countably many of the components of the Julia set of a hyperbolic meromorphic function Jordan arcs or single points?

5. A continuous map $T : \widehat{X} \to \widehat{X}$ of a compact metric space $(\widehat{X}, d)$ is called expansive if there exists a $\delta > 0$ such that if for $x, y \in \widehat{X}$ and for each $n \geq 0$, $d(T^n(x), T^n(y)) < \delta$, then $x = y$.

This definition of an expansive self-mapping of compact metric space is neither suitable for the case when T is a continuous map from X_0 into $\widehat{X}$, where X_0 is a dense open subset of $\widehat{X}$, nor the case when T is an infinite-to-one continuous map of $\widehat{X}$ from X_0. Therefore, we modify the definition ([14]).

Definition 8.1. A continuous map $T : X_0 \to \widehat{X}$ is called precisely expansive if there exists a $\delta > 0$ such that for $x \neq y$ in $\widehat{X}$, one of the following statements holds

(1) for some $s \geq 0$, at least one of $T^s(x)$ and $T^s(y)$ is in $\widehat{X} \setminus X_0$ and $T^s(x) \neq T^s(y)$;

(2) for some $m \geq 1$ with $T^m(x) = T^m(y) \in \widehat{X}$ but $T^{m-1}(x) \neq T^{m-1}(y)$, we have $y \notin T_x^{-m}(B(T^m(x), \delta))$ and $x \notin T_y^{-m}(B(T^m(y), \delta))$;

(3) for a sequence of natural numbers $\{n_k\}$ with $n_k < n_{k+1} \to \infty$,

$$y \notin T_x^{-n_k}(B(T^{n_k}(x), \delta)) \text{ and } x \notin T_y^{-n_k}(B(T^{n_k}(y), \delta)).$$

Let φ be a continuous function on $\hat{X}$ and summable, i.e., $\sup_{x \in \hat{X}} \sum_{T(y)=x} e^{\varphi(y)} < \infty$.

Problem 5. Does T have an atomless $\exp(-\varphi + P(T, \varphi))$-conformal measure μ and a μ-equivalent invariant measure ν if $T : X_0 \to \hat{X}$ is precisely expansive with expansive constant $\delta > 0$ and the Bowen condition

$$\left| \sum_{k=0}^{n-1} \varphi(T^k(y)) - \sum_{k=0}^{n-1} \varphi(T^k(y')) \right| \leq K$$

holds for $\forall n \in \mathbb{N}$, a constant $K > 0$ and y, y' with the Bowen distance $d_n(y, y') < \delta$?

Aaronson, Denker and Urbanski ([1],[2] and [3]) proved that an expansive rational function on its Julia set has an atomless s-conformal measure μ and s is the Poincare exponent and μ-equivalent invariant measure.

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