Note Title 15/05/2013

Applications of harmonic functions in complex dynamics

$f: \mathbb{C} \to \mathbb{C}$ transcendental entire function throughout

1. Multiply connected Fatou components

U a m-conn Fatou component of f, $U_n = f^n(U)$

Theorem (Baker 1984)
- U_n is bounded for all n
- U_{n+1} surrounds U_n for all n
- $\operatorname{dist}(0, U_n) \to \infty$ as $n \to \infty$

Exs Baker, Hinkkanen, Kisaka + Shishikura, Bergweiler, Osborne, Bishop

Theorem (Zheng, 2006) For large n,

$$U_n \supset A(r_n, R_n), \text{ where } R_n/r_n \to \infty \text{ as } n \to \infty.$$

Theorem 1 (Bergweiler + R + Stallard) Let $z_0 \in U$.

(a) $\exists\, \varepsilon > 0$ such that, for large n,

$$U_n \supset A\left(|f^n(z_0)|^{1-\varepsilon}, |f^n(z_0)|^{1+\varepsilon}\right).$$

(b) $$h(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|}, \quad z \in U,$$

is a positive harmonic function.

(c) Suppose $A\left(|f^n(z_0)|^{a_n}, |f^n(z_0)|^{b_n}\right) \subset U_n$ is maximal. Then

$a_n \to a$ and $b_n \to b$, with $0 \le a < 1 < b \le \infty$,

and

$$h(z) = \begin{cases} a, & z \in \partial U \setminus \partial_{out} U, \\ b, & z \in \partial_{out} U. \end{cases}$$

Care: $b = \infty$ case.

Covering lemmas

Lemma 1 (Covering using hyperbolic metric) $\exists\, \delta > 0$ s.t. if

- f omits 0 in $A(R, R')$
- $\rho_{A(R,R')}(z_1, z_2) < \delta$ and $|f(z_2)| \geq 2|f(z_1)|$

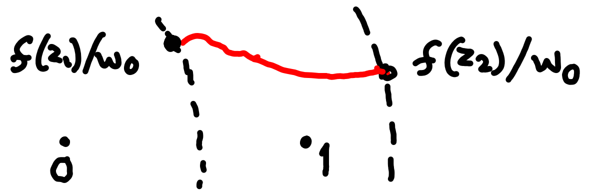
then

$$f(A(R,R')) \supset \bar{A}(|f(z_1)|, |f(z_2)|).$$

Proof Suppose f omits $w_0 \in \bar{A}(|f(z_1)|, |f(z_2)|)$. Then

$$\delta \geq \delta_{A(R,R')}(z_1, z_2) \geq \rho_{\mathbb{C}\setminus\{0,w_0\}}(f(z_1), f(z_2))$$

$$= \rho_{\mathbb{C}\setminus\{0,1\}}(f(z_1)/w_0, f(z_2)/w_0). \quad ※$$



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Lemma 2 (Using Harnack's inequality) Suppose $r > 0$, $0 < a < b$,

$$m(\rho, f) = \min_{|z|=\rho} |f(z)| > 1, \quad \text{for } r^a < \rho < r^b.$$

If $\delta = \dfrac{1}{\sqrt{\log r}} < \min\left\{1, \dfrac{b-a}{4\pi}\right\}$, then

$$m(\rho, f) \geq M(\rho, f)^{1-\delta}, \quad \text{for } r^{a+2\pi\delta} \leq \rho \leq r^{b+2\pi\delta}.$$

Proof $u(z) = \log|f(z)|$ is positive harmonic in $A(r^a, r^b)$

so

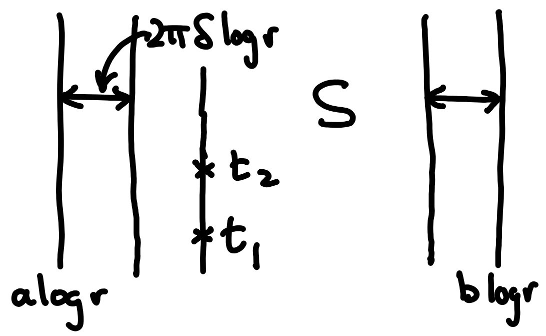
$U(t) = u(e^t)$ is positive harmonic in $S = \{t : a\log r < \operatorname{Re} t < b \log r\}$.

Use Harnack inequality:

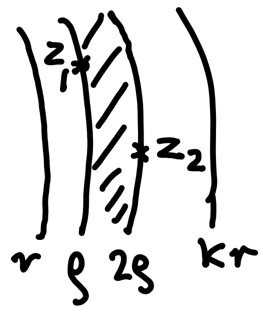
$\operatorname{Re} t_1 = \operatorname{Re} t_2$, $|\operatorname{Im} t_1 - \operatorname{Im} t_2| \leq \pi$

$$\frac{2\pi\sqrt{\log r} - \pi}{2\pi\sqrt{\log r} + \pi} \leq \frac{U(t_2)}{U(t_1)} \leq \frac{2\pi\sqrt{\log r} + \pi}{2\pi\sqrt{\log r} - \pi}$$

$$1 - \delta \leq \frac{1 - \frac{1}{2}\delta}{1 + \frac{1}{2}\delta} \leq \frac{u(z_2)}{u(z_1)}. \quad ■$$



<u>Outline proof of Theorem 1(a)(b)</u> By Zheng's theorem, can take $K>0$ and $N\in\mathbb{N}$ so large that



- $\rho_{A(r,Kr)}(z_1,z_2)<\delta$, for $r>0$, $z_1,z_2\in\bar{A}(\rho,2\rho)$, $\rho=\sqrt{K}r$
- $U_N\supset A(r,Kr)$, $r=r(N)$
- $M(2\rho)\geq 2M(\rho)$, $M(\rho)\geq\rho^p$, $2^p\geq K$.

Let $2\rho=\rho^c$. So $c>1$, $\rho^{c-1}=2$.

Take $M(\rho)=|f(z_1)|$, $M(2\rho)=|f(z_2)|$. By Lemma 1,

$$
\begin{aligned}
U_{N+1} &\supset f(A(r,Kr))\\
&\supset A(M(\rho),M(2\rho))\\
&\supset A(M(\rho),M(\rho)^c)\\
&\supset A(M(\rho),KM(\rho)).
\end{aligned}
$$

$M(\rho^c)\geq M(\rho)^c$

$\geq KM(\rho)$

Since

$M(\rho)^{c-1}\geq(\rho^p)^{c-1}=2^p\geq K$

We can now apply Lemma 1 again, and repeatedly to give

$$U_n\supset f^n(A(r,Kr))\supset A(R_n,R_n^c),\quad \text{where } R_n\to\infty.$$

So U_n contain 'fat' annuli, but what is the dynamical behaviour of iterates in these?

Lemma 2 (Harnack inequality) gives

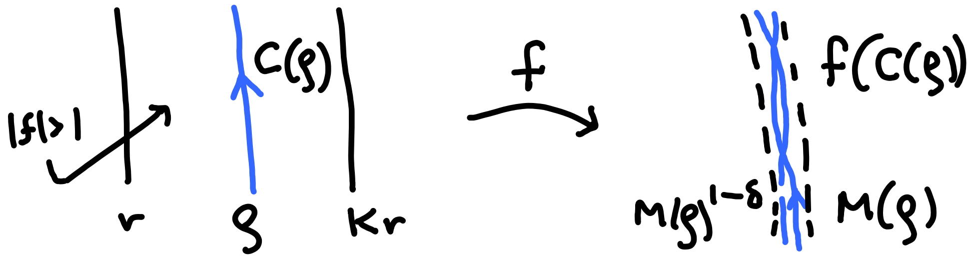


Image of a circle winds round 0 in a 'thin' annulus.

Roughly:

if $|f|>1$ in $A(r, r^c)$ and $r<\rho<r^c$, then

$$f(C(\rho)) \simeq C(M(\rho)) \quad \text{and} \quad f(A(r,r^c)) \simeq A(M(r), M(r)^{c'}).$$

where $c' \geq c$. More precisely, $\exists$ $r_n > 0$ and $c_n > 1$ s.t.

$$U_n \supset \underbrace{A(r_n, r_n^{c_n})}_{A_n} \supset \underbrace{A\left(r_n^{1+6\pi\delta_n}, r_n^{c_n(1-6\pi\delta_n)}\right)}_{A_n'}$$

$\delta_n = \dfrac{1}{\sqrt{\log r_n}}$

with

$$f(A_n) \supset A_{n+1} \quad \text{and} \quad f(A_n') \subset A_{n+1}'.$$

Take a base point $z_0 \in A_0'$ and put

$$h_n(z) = \frac{\log|f^n(z)|}{\log|f^n(z_0)|} \quad \text{positive harmonic in } U_0 \text{ and } h_n(z_0)=1.$$

By Harnack's theorem, $\exists$ a convergent subsequence

$$\lim_{k\to\infty} h_{n_k}(z) = h(z), \quad \text{for } z \in U_0.$$

Since

$$|f^n(z)| = |f^n(z_0)|^{h_n(z)}$$

and

$$|f^n(z_0)| \simeq M^n(|z_0|)$$

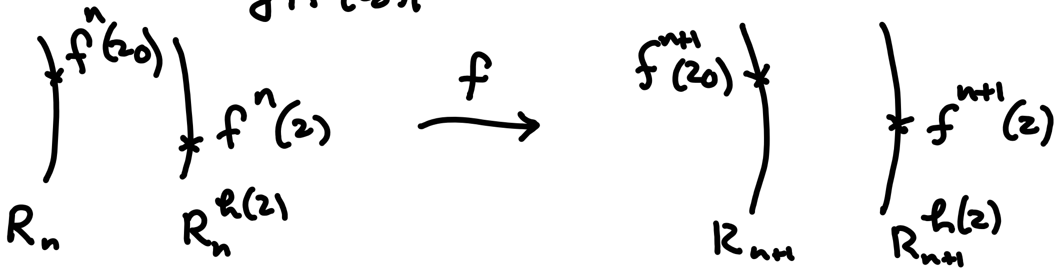
we deduce convergence of h_n to h by using Hadamard convexity. ∎

2. Sufficient conditions for logarithmic regularity

Let U be a m-conn Fatou component of f, $U_n = f^n(U)$.

Within the maximal annulus $B_0 \subset U_0 = U$, we have

$$h(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|} \quad \text{so} \quad |f^n(z)| \simeq |f^n(z_0)|^{h(z)}.$$



so

$$M(R_n^{h(z)}) \simeq M(R_n)^{h(z)},$$ 'equality' in Hadamard convexity.

Not log-regular!

Log-regular $\exists\ r_0 > 0,\ k > 1,\ d > 1$ s.t.

$$M(r^k) \geq M(r)^{kd}, \quad \text{for } r \geq r_0$$

$\Leftrightarrow$ $\exists\ t_0 > 0$ and $\varepsilon > 0$ s.t.

$$\frac{\varphi'(t)}{\varphi(t)} \geq \frac{1+\varepsilon}{t}, \quad \text{for } t \geq t_0.$$

- Introduced by Anderson + Hinkkanen (1998) in relation to Baker's conjecture.
  Equivalent condition used by Wang (2001).

- Used and investigated by R & S (2013) and Sixsmith (2011) in relation to the existence of fast escaping spiders' webs.

- Used by R & S in relation to equality of subsets of $I(f)$.
  Implies: 'Quite fast' escaping points are fast escaping.

Theorem 2 (R+S, 2013) If
$$m(r) \le M(r)^{1-K/\log r}, \quad r \ge r_0,$$
where $K = 4\log 4$, then f is log-regular.

In particular, this holds if

- f is in class $\mathcal{B}$, since f is then bounded on a path to ∞
- f has an unbounded periodic Fatou component, since this implies $m(r) = O(r)$ as $r \to \infty$ by a theorem of Zheng.

Proof of Theorem 2 is based on a harmonic measure estimate of Beurling (1933) related to the Milloux–Schmidt inequality. Let $\mathbb{D}$ be the open unit disc.

Lemma 3 Let E be a closed subset of $\overline{\mathbb{D}}$ and E^* the circular projection of E on the negative real axis. Then, for $0 < r < 1$, $|z| = r$,
$$\omega(z, \partial\mathbb{D}, \mathbb{D}\setminus E) \le \omega(r, \partial\mathbb{D}, \mathbb{D}\setminus E^*)$$
$$< \frac{4\sqrt{2}}{\pi}\exp\left(-\frac{1}{2}\int_{-E^*\cap[r,1)} \frac{dt}{t}\right)$$

So if f is analytic in $\{z : |z| < r_0\}$ and
$$E = \{t \in (r_1, r_2) : m(t) \le \mu\}, \quad \text{where } 0 < \mu < M(r_1),$$
then
$$\log\frac{M(r_2)}{\mu} \ge \frac{\pi}{4\sqrt{2}}\exp\left(\frac{1}{2}\int_E \frac{dt}{t}\right)\log\frac{M(r_1)}{\mu}.$$

Use this to prove Theorem 2 - easy if f bdd on a path!

3. Boundaries of escaping Fatou components

Question If U is a Fatou component in $I(f)$ is
$$\partial U \cap I(f) \neq \emptyset \,?$$

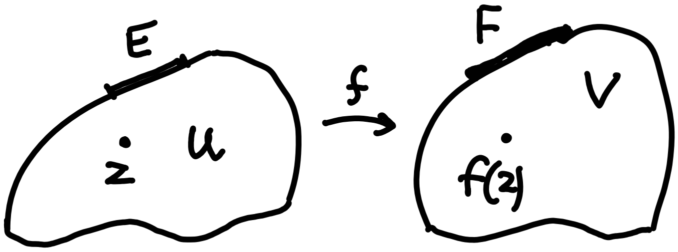
- In general $\partial U \not\subset I(f)$ e.g. $f(z) = z + 1 + e^{-z}$
- If U is a Fatou component in $A(f)$, then $\partial U \subset A(f)$.

Theorem 3 (R+S, 2011) If U is a WD and $U \subset I(f)$, then
$$\partial U \cap I(f) \neq \emptyset.$$
Indeed
$$\partial U \cap I(f)^c \text{ has zero harmonic measure in } U.$$

What if U is a Baker domain?

Lemma 4 If U, V are domains with $E \subset \partial U$, $F \subset \partial V$, and f is analytic in U and continuous in $U \cup E$, with
$$f(U) \subset V \quad \text{and} \quad f(E) \subset F,$$
then
$$\omega(f(z), F, V) \geq \omega(z, E, U).$$

Löwner's lemma

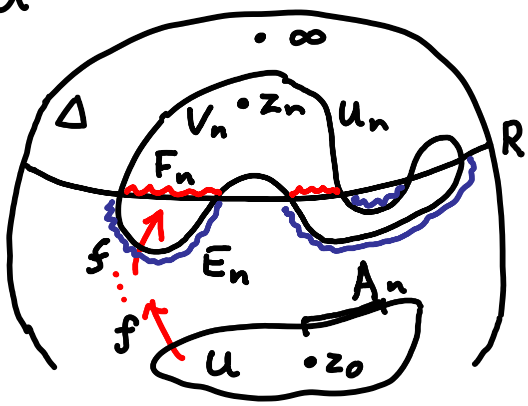


Proof of Theorem 3

Suppose U is an escaping WD. Fix $z_0 \in U$ and define $z_n = f^n(z_0)$, $f^n(U) \subset U_n$ a Fatou component.

Let $R>0$ and put
$$A_n = A_n(R) = \{ z\in \partial U : |f^n(z)| \le R \}, \qquad n\in \mathbb{N}.$$

Need to show $\overline{\lim_{n\to\infty}} A_n$ has harmonic measure 0 w.r.t. U.
$$\overline{\lim_{n\to\infty}} A_n = \{ z\in\partial U : |f^n(z)|\le R, \text{ inf. many } n\} = \bigcap_{m=1}^{\infty} \bigcup_{n\ge m} A_n .$$
Choose $N=N(R)$ s.t. $|z_n| > 2R$, for $n\ge N$. Put



$\Delta = \{z : |z|>R\}\cup\{\infty\}$
$E_n = \partial U_n \cap \{z : |z|\le R\}$
V_n is component of $U_n\cap\Delta$ containing z_n
$F_n = \partial V_n \cap \{z : |z| = R\}$.

Then
$$\omega(z, E_n, U_n) \le \omega(z, F_n, \Delta), \quad z\in V_n,$$
maximum principle

so
$$\sum_{n\ge N}\omega(z_n, E_n, U_n) \le \sum_{n\ge N}\omega(z_n, F_n, \Delta) \le 3\sum_{n\ge N}\omega(\infty, F_n,\Delta) \le 3.$$
Harnack

Now $f^n(U)\subset U_n$ and $f^n(A_n) = E_n$, so
$$\sum_{n\ge N}\omega(z_0, A_n, U) \le \sum_{n\ge N}\omega(z_n, E_n, U_n) < \infty,$$
Lemma 4

$$\Rightarrow \quad \omega\Big(z_0, \bigcup_{n\ge m} A_n, U\Big) \le \sum_{n\ge m}\omega(z_0, A_n, U) \to 0 \quad \text{as } m\to\infty,$$

$$\Rightarrow \quad \omega\Big(z_0, \overline{\lim_{n\to\infty}} A_n, U\Big) = \lim_{m\to\infty}\omega\Big(z_0, \bigcup_{n\ge m}A_n, U\Big) = 0. \qquad \blacksquare$$

Cor If G is a bounded S-conn. domain, then
$$G\cap I(f)\ne\emptyset \quad\Rightarrow\quad \partial G\cap I(f)\ne\emptyset.$$
Hence
$$I(f)\cup\{\infty\} \text{ is connected.}$$