The size of the Julia set of some families of functions outside the Eremenko-Lyubich class

Dave Sixsmith

Department of Mathematics and Statistics
The Open University

ICMS
23rd May 2013

Definitions

Definition

The Eremenko-Lyubich class, $\mathcal{B}$, is the set of transcendental entire functions for which the set of singular values is bounded.

Definitions

Definition

The Eremenko-Lyubich class, $\mathcal{B}$, is the set of transcendental entire functions for which the set of singular values is bounded.

Definition

For each $n \in \mathbb{N}$, let $\omega_n = \exp(2\pi i/n)$ be an nth root of unity. Define a family of transcendental entire functions,

$$S_n = \{f : f(z) = \sum_{k=0}^{n-1} a_k \exp(\omega_n^k z), \quad \text{where each } a_k \neq 0\}.$$

Definitions

Definition

The Eremenko-Lyubich class, $\mathcal{B}$, is the set of transcendental entire functions for which the set of singular values is bounded.

Definition

For each $n \in \mathbb{N}$, let $\omega_n = \exp(2\pi i/n)$ be an nth root of unity. Define a family of transcendental entire functions,

$$S_n = \{f : f(z) = \sum_{k=0}^{n-1} a_k \exp(\omega_n^k z), \quad \text{where each } a_k \neq 0\}.$$

Observe that $S_1 \cup S_2 \subset \mathcal{B}$, but $S_n \cap \mathcal{B} = \emptyset$, for $n \geq 3$.

Examples

The family S_1 is the exponential family

$$\{f : f(z) = \lambda \exp(z),\ \lambda \neq 0\}.$$

Examples

The family S_1 is the exponential family

$$\{f : f(z) = \lambda \exp(z),\ \lambda \neq 0\}.$$

The family S_2 consists of functions of the form

$$f(z) = \gamma \exp(z) + \delta \exp(-z),\ \gamma \neq 0,\ \delta \neq 0,$$

and so, up to a conjugacy, is the same as the cosine family.

Examples

The family S_1 is the exponential family

$$\{f : f(z) = \lambda \exp(z),\ \lambda \neq 0\}.$$

The family S_2 consists of functions of the form

$$f(z) = \gamma \exp(z) + \delta \exp(-z),\ \gamma \neq 0,\ \delta \neq 0,$$

and so, up to a conjugacy, is the same as the cosine family.

An example of a function in S_4 is

$$f(z) = \frac{1}{2}\left(e^z + e^{iz} + e^{-z} + e^{-iz}\right) = \cos z + \cosh z.$$

S_1 and S_2

Theorem (McMullen 1987)

(i) *If $f \in S_1$, then $\dim_H J(f) \cap A(f) = 2$.*

(ii) *There exist functions $f \in S_1$ such that $J(f)$ has area zero.*

(iii) *If $f \in S_2$, then $J(f) \cap A(f)$ has positive area.*

S_1 and S_2

Theorem (McMullen 1987)

(i) *If* $f \in S_1$*, then* $\dim_H J(f) \cap A(f) = 2$*.*

(ii) *There exist functions* $f \in S_1$ *such that* $J(f)$ *has area zero.*

(iii) *If* $f \in S_2$*, then* $J(f) \cap A(f)$ *has positive area.*

Note that McMullen stated his results (i) and (iii) for $J(f)$, but it follows easily from his proof that this stronger result holds.

A generalisation of McMullen's result

Theorem (Sixsmith 2013)

If $f \in S_n$, for $n \geq 3$, then $J(f) \cap A(f)$ has positive area.

A generalisation of McMullen's result

Theorem (Sixsmith 2013)

If $f \in S_n$, for $n \geq 3$, then $J(f) \cap A(f)$ has positive area.

- It does not seem possible to use Koebe's distortion theorem, and so the proof uses a distortion result of McMullen.

Spiders' webs

By results of Schleicher and Zimmer (2003), Rottenfusser and Schleicher (2008), and Rempe, Rippon and Stallard (2010) we have, for $f \in S_1 \cup S_2$, that $J(f) \cap A(f)$ is a collection of curves – sometimes connected and sometimes disconnected.

Spiders' webs

By results of Schleicher and Zimmer (2003), Rottenfusser and Schleicher (2008), and Rempe, Rippon and Stallard (2010) we have, for $f \in S_1 \cup S_2$, that $J(f) \cap A(f)$ is a collection of curves – sometimes connected and sometimes disconnected.

Theorem (Sixsmith 2013)

If $f \in S_n$, for $n \geq 3$, then each of the following is a spider's web;

$$A_R(f),\ A(f),\ I(f),\ A_R(f) \cap J(f),\ A(f) \cap J(f),\ I(f) \cap J(f),\ \textit{and}\ \ J(f).$$

Spiders' webs

By results of Schleicher and Zimmer (2003), Rottenfusser and Schleicher (2008), and Rempe, Rippon and Stallard (2010) we have, for $f \in S_1 \cup S_2$, that $J(f) \cap A(f)$ is a collection of curves – sometimes connected and sometimes disconnected.

Theorem (Sixsmith 2013)

If $f \in S_n$, for $n \geq 3$, then each of the following is a spider's web;

$$A_R(f),\ A(f),\ I(f),\ A_R(f) \cap J(f),\ A(f) \cap J(f),\ I(f) \cap J(f),\ \textit{and}\ \ J(f).$$

Note that $f \in \mathcal{B}$ implies that $A_R(f)$ is not a spider's web.

Further Research

Rippon and Stallard (2012) studied functions of the form

$$f(z) = cz^p \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right), \quad \text{for } c \in \mathbb{R}\backslash\{0\},\ a_k \in \mathbb{R}^+. \qquad (1)$$

Further Research

Rippon and Stallard (2012) studied functions of the form

$$f(z) = cz^p \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right), \quad \text{for } c \in \mathbb{R}\backslash\{0\},\ a_k \in \mathbb{R}^+. \qquad (1)$$

- These functions are not, in general, in class $\mathcal{B}$.

Further Research

Rippon and Stallard (2012) studied functions of the form

$$f(z) = cz^p \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right), \quad \text{for } c \in \mathbb{R}\backslash\{0\},\ a_k \in \mathbb{R}^+. \tag{1}$$

- These functions are not, in general, in class $\mathcal{B}$.
- We hope to show that, subject to a growth condition, functions of a form more general than (1) either have a multiply connected Fatou component, or satisfy $dim_H J(f) \cap A(f) = 2$.

Further Research

Rippon and Stallard (2012) studied functions of the form

$$f(z) = cz^p \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right), \quad \text{for } c \in \mathbb{R}\backslash\{0\},\ a_k \in \mathbb{R}^+. \qquad (1)$$

- These functions are not, in general, in class $\mathcal{B}$.
- We hope to show that, subject to a growth condition, functions of a form more general than (1) either have a multiply connected Fatou component, or satisfy $dim_H J(f) \cap A(f) = 2$.
- Proof uses a technique from Bergweiler and Karpińska *'On the Hausdorff dimension of the Julia set of a regularly growing entire function'*, (2010).