Applications of the hyperbolic metric in complex dynamics

Gwyneth Stallard

The Open University

ICMS, Edinburgh
May 2013

Basic definitions

Definition

The **Fatou set** (or stable set) is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighbourhood of } z\}.$$

Basic definitions

Definition

The **Fatou set** (or stable set) is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighbourhood of } z\}.$$

Definition

The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

Basic definitions

Definition

The **Fatou set** (or stable set) is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighbourhood of } z\}.$$

Definition

The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

Definition

The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Subsets of the escaping set

Definition

The **fast escaping set** is

Subsets of the escaping set

Definition

The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \; \forall \; n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

Subsets of the escaping set

Definition

The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \ \forall\, n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

Definition

The **slow escaping set** is

$$L(f) = \{z \in I(f) : \limsup_{n \to \infty} \frac{1}{n} \log |f^n(z)| < \infty\}.$$

Subsets of the escaping set

Definition

The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \; \forall \; n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

Definition

The **slow escaping set** is

$$L(f) = \{z \in I(f) : \limsup_{n \to \infty} \frac{1}{n} \log |f^n(z)| < \infty\}.$$

Eremenko (1989) showed that $J(f) = \partial I(f)$,

Subsets of the escaping set

Definition

The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \ \forall\, n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

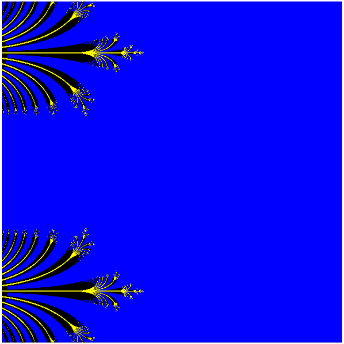
Definition

The **slow escaping set** is

$$L(f) = \{z \in I(f) : \limsup_{n \to \infty} \frac{1}{n} \log |f^n(z)| < \infty\}.$$

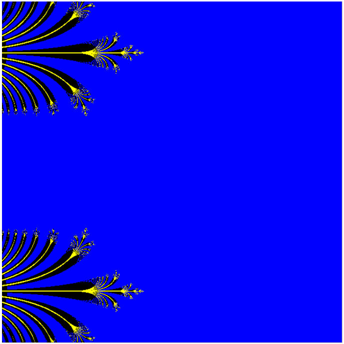
Eremenko (1989) showed that $J(f) = \partial I(f)$,
and conjectured that all the components of $I(f)$ are unbounded.

Examples
Fatou's function



$$f(z) = z + 1 + e^{-z}$$

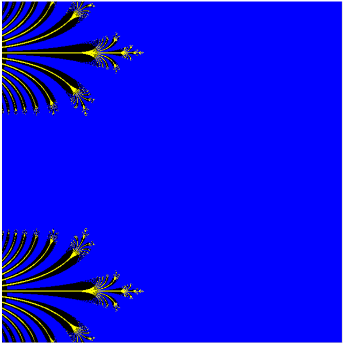
Examples
Fatou's function



$f(z) = z + 1 + e^{-z}$

- $F(f)$ is a Baker domain – a periodic Fatou component in $I(f)$ (actually in $L(f)$)

Examples
Fatou's function



$f(z) = z + 1 + e^{-z}$

- $F(f)$ is a Baker domain – a periodic Fatou component in $I(f)$ (actually in $L(f)$)
- $J(f)$ is a Cantor bouquet of curves – all points apart from endpoints are in $A(f)$

A Fatou component U is a **wandering domain** if

$$f^n(U) \cap f^m(U) = \emptyset \text{ for } n \neq m.$$

A Fatou component U is a **wandering domain** if

$$f^n(U) \cap f^m(U) = \emptyset \text{ for } n \neq m.$$

Theorem (Baker, 1984)

If U is a multiply connected Fatou component then

- *U is a wandering domain in $I(f)$*
- *$f^{n+1}(U)$ surrounds $f^n(U)$ for large n.*

Examples

Multiply connected wandering domains

A Fatou component U is a **wandering domain** if

$$f^n(U) \cap f^m(U) = \emptyset \text{ for } n \neq m.$$

Theorem (Baker, 1984)

If U is a multiply connected Fatou component then

- *U is a wandering domain in $I(f)$*
- *$f^{n+1}(U)$ surrounds $f^n(U)$ for large n.*

Theorem (Rippon and Stallard, 2005)

If U is a multiply connected Fatou component then

- *$U \subset A(f)$*
- *$I(f)$ is connected.*

Application 1
Speed of escape in Fatou components

Key distortion property:

Theorem (Baker, 1988)

Let U be a Fatou component in $I(f)$ and let K be a compact subset of U.

- *There exist $C > 1$, $N \in \mathbb{N}$ such that*

$$|f^n(z_0)| \leq |f^n(z_1)|^C, \text{ for } z_0, z_1 \in K, \; n \geq N.$$

Application 1

Speed of escape in Fatou components

Key distortion property:

Theorem (Baker, 1988)

Let U be a Fatou component in $I(f)$ and let K be a compact subset of U.

- *There exist $C > 1$, $N \in \mathbb{N}$ such that*

$$|f^n(z_0)| \leq |f^n(z_1)|^C, \text{ for } z_0, z_1 \in K, \ n \geq N.$$

- *If, in addition, U is simply connected, then there exist $C > 1$, $N \in \mathbb{N}$ such that*

$$|f^n(z_0)| \leq C|f^n(z_1)|, \text{ for } z_0, z_1 \in K, \ n \geq N.$$

Proof of distortion property

(a) Let Ω denote the plane punctured at 2 points in $J(f)$ and let $z_0, z_1 \in U$. Then, for large $n \in \mathbb{N}$,

$$\rho_U(z_0, z_1) \geq \rho_{f^n(U)}(f^n(z_0), f^n(z_1))$$

Proof of distortion property

(a) Let Ω denote the plane punctured at 2 points in $J(f)$ and let $z_0, z_1 \in U$. Then, for large $n \in \mathbb{N}$,

$$\rho_U(z_0, z_1) \geq \rho_{f^n(U)}(f^n(z_0), f^n(z_1)) \geq \rho_\Omega(f^n(z_0), f^n(z_1))$$

Proof of distortion property

(a) Let Ω denote the plane punctured at 2 points in $J(f)$ and let $z_0, z_1 \in U$. Then, for large $n \in \mathbb{N}$,

$$\rho_U(z_0, z_1) \geq \rho_{f^n(U)}(f^n(z_0), f^n(z_1)) \geq \rho_\Omega(f^n(z_0), f^n(z_1))$$

$$\geq c \int_{|f^n(z_0)|}^{|f^n(z_1)|} \frac{dt}{t \log t}$$

Proof of distortion property

(a) Let Ω denote the plane punctured at 2 points in $J(f)$ and let $z_0, z_1 \in U$. Then, for large $n \in \mathbb{N}$,

$$\rho_U(z_0, z_1) \geq \rho_{f^n(U)}(f^n(z_0), f^n(z_1)) \geq \rho_\Omega(f^n(z_0), f^n(z_1))$$

$$\geq c \int_{|f^n(z_0)|}^{|f^n(z_1)|} \frac{dt}{t \log t} = c \log\left(\frac{\log|f^n(z_1)|}{\log|f^n(z_2)|}\right).$$

Proof of distortion property

(a) Let Ω denote the plane punctured at 2 points in $J(f)$ and let $z_0, z_1 \in U$. Then, for large $n \in \mathbb{N}$,

$$\rho_U(z_0, z_1) \geq \rho_{f^n(U)}(f^n(z_0), f^n(z_1)) \geq \rho_\Omega(f^n(z_0), f^n(z_1))$$

$$\geq c \int_{|f^n(z_0)|}^{|f^n(z_1)|} \frac{dt}{t \log t} = c \log \left(\frac{\log |f^n(z_1)|}{\log |f^n(z_2)|} \right).$$

(b) follows similarly using $\Omega = \mathbb{C} \setminus \gamma$, where γ is an unbounded continuum.

Corollaries of distortion property

Corollary (Baker, 1988)

If U is an invariant Baker domain then

- *There exists a path $\Gamma \subset U$ and $C > 1$ such that*

$$|z|/C < |f(z)| < C|z|, \quad \text{for } z \in \Gamma.$$

Corollaries of distortion property

Corollary (Baker, 1988)

If U is an invariant Baker domain then

- *There exists a path $\Gamma \subset U$ and $C > 1$ such that*

$$|z|/C < |f(z)| < C|z|, \ \text{for } z \in \Gamma.$$

- $U \subset L(f)$.

Corollaries of distortion property

Corollary (Baker, 1988)

If U is an invariant Baker domain then

- *There exists a path $\Gamma \subset U$ and $C > 1$ such that*

$$|z|/C < |f(z)| < C|z|, \text{ for } z \in \Gamma.$$

- $U \subset L(f)$.

Remarks 1. $\Gamma = \bigcup_{n\geq 0} f^n(\Gamma_0)$, where Γ_0 joins $z_0 \in U$ to $f(z_0)$.

Corollaries of distortion property

Corollary (Baker, 1988)

If U is an invariant Baker domain then

- *There exists a path $\Gamma \subset U$ and $C > 1$ such that*

$$|z|/C < |f(z)| < C|z|, \text{ for } z \in \Gamma.$$

- $U \subset L(f)$.

Remarks 1. $\Gamma = \bigcup_{n \geq 0} f^n(\Gamma_0)$, where Γ_0 joins $z_0 \in U$ to $f(z_0)$.
2. ∂U is not necessarily in $I(f)$.

Corollaries of distortion property

Corollary (Baker, 1988)

If U is an invariant Baker domain then

- *There exists a path $\Gamma \subset U$ and $C > 1$ such that*

$$|z|/C < |f(z)| < C|z|, \text{ for } z \in \Gamma.$$

- $U \subset L(f)$.

Remarks 1. $\Gamma = \bigcup_{n \geq 0} f^n(\Gamma_0)$, where Γ_0 joins $z_0 \in U$ to $f(z_0)$.
2. ∂U is not necessarily in $I(f)$.

Corollary (Rippon and Stallard, 2005)

If U is a simply connected wandering domain in $A_R(f)$, then $\overline{U} \subset A_R(f)$.

Application 2
Geometry and speed of escape

Basic hyperbolic metric estimates were used to prove the following:

Basic hyperbolic metric estimates were used to prove the following:

Theorem (Rippon, 2006)

Let U be an invariant Baker domain and let Γ be path in U constructed as before.

Basic hyperbolic metric estimates were used to prove the following:

Theorem (Rippon, 2006)

Let U be an invariant Baker domain and let Γ be path in U constructed as before.
Then there exist $C, c, c' > 0$ such that, for $z \in \Gamma$,

- $dist(z, \partial U) \leq C \frac{|f(z)-z|}{\rho_U(f(z),z)}$

Basic hyperbolic metric estimates were used to prove the following:

Theorem (Rippon, 2006)

Let U be an invariant Baker domain and let Γ be path in U constructed as before.
Then there exist $C, c, c' > 0$ such that, for $z \in \Gamma$,

- $dist(z, \partial U) \leq C\frac{|f(z)-z|}{\rho_U(f(z),z)}$
- $dist(z, \partial U) \geq c\frac{|f(z)-z|}{\rho_U(f(z),z)} \geq c'|f(z) - z|.$

Basic hyperbolic metric estimates were used to prove the following:

Theorem (Rippon, 2006)

Let U be an invariant Baker domain and let Γ be path in U constructed as before.
Then there exist $C, c, c' > 0$ such that, for $z \in \Gamma$,

- $dist(z, \partial U) \leq C\frac{|f(z)-z|}{\rho_U(f(z),z)}$
- $dist(z, \partial U) \geq c\frac{|f(z)-z|}{\rho_U(f(z),z)} \geq c'|f(z) - z|.$

So, the thinner the Baker domain U, the closer f is to the identity in U.

Construction of examples

- The thinner the Baker domain U, the closer f is to the identity in U.

Construction of examples

- The thinner the Baker domain U, the closer f is to the identity in U.
- These ideas were used by Rempe and Rippon (2012) together with approximation theory to construct Baker domains with arbitrarily slow rate of escape.

Construction of examples

- The thinner the Baker domain U, the closer f is to the identity in U.
- These ideas were used by Rempe and Rippon (2012) together with approximation theory to construct Baker domains with arbitrarily slow rate of escape.
- Hyperbolic metric estimates plus approximation theory are often used to construct examples with specified properties (see papers of Rempe et. al.).

Application 3

Speed of escape in the Julia set

Take $R > 0$ such that $M^n(R) \to \infty$ as $n \to \infty$ and let

$$A_0(R) = \{z : |z| < R\}, \; A_n(R) = \{z : M^{n-1}(R) \leq |z| < M^n(R)\}.$$

Application 3
Speed of escape in the Julia set

Take $R > 0$ such that $M^n(R) \to \infty$ as $n \to \infty$ and let

$$A_0(R) = \{z : |z| < R\}, \; A_n(R) = \{z : M^{n-1}(R) \leq |z| < M^n(R)\}.$$

Theorem (Rippon and Stallard)

There exists $R > 0$ and

$$B_n = \{z : r_n \leq |z| \leq r'_n\} \subset A_n(R)$$

such that

Application 3

Speed of escape in the Julia set

Take $R > 0$ such that $M^n(R) \to \infty$ as $n \to \infty$ and let

$A_0(R) = \{z : |z| < R\},\ A_n(R) = \{z : M^{n-1}(R) \leq |z| < M^n(R)\}.$

Theorem (Rippon and Stallard)

There exists $R > 0$ and

$$B_n = \{z : r_n \leq |z| \leq r_n'\} \subset A_n(R)$$

such that

- $B_n \cap J(f) \neq \emptyset$, *for* $n \in \mathbb{N}$;

Application 3

Speed of escape in the Julia set

Take $R > 0$ such that $M^n(R) \to \infty$ as $n \to \infty$ and let

$$A_0(R) = \{z : |z| < R\}, \; A_n(R) = \{z : M^{n-1}(R) \leq |z| < M^n(R)\}.$$

Theorem (Rippon and Stallard)

There exists $R > 0$ and

$$B_n = \{z : r_n \leq |z| \leq r'_n\} \subset A_n(R)$$

such that

- $B_n \cap J(f) \neq \emptyset$, *for* $n \in \mathbb{N}$;
- $f(B_n) \supset B_{n+1}$, *for* $n \in \mathbb{N}$;

Take $R > 0$ such that $M^n(R) \to \infty$ as $n \to \infty$ and let

$$A_0(R) = \{z : |z| < R\}, \; A_n(R) = \{z : M^{n-1}(R) \leq |z| < M^n(R)\}.$$

Theorem (Rippon and Stallard)

There exists $R > 0$ and

$$B_n = \{z : r_n \leq |z| \leq r'_n\} \subset A_n(R)$$

such that

- $B_n \cap J(f) \neq \emptyset$, *for* $n \in \mathbb{N}$;
- $f(B_n) \supset B_{n+1}$, *for* $n \in \mathbb{N}$;
- *there exists a subsequence* B_{n_j} *such that*

$$f(B_{n_j}) \supset B_n, \; \textit{for } 0 \leq n \leq n_j \textit{ with at most 1 exception.}$$

Speed of escape in the Julia set

By using this sequence of annuli B_n, we can construct points in $J(f)$ with various "annular itineraries" including

Speed of escape in the Julia set

By using this sequence of annuli B_n, we can construct points in $J(f)$ with various "annular itineraries" including

- points in $A_R(f)$;

Speed of escape in the Julia set

By using this sequence of annuli B_n, we can construct points in $J(f)$ with various "annular itineraries" including

- points in $A_R(f)$;
- points in $L(f)$;

Speed of escape in the Julia set

By using this sequence of annuli B_n, we can construct points in $J(f)$ with various "annular itineraries" including

- points in $A_R(f)$;
- points in $L(f)$;
- points that escape arbitrarily slowly;

Speed of escape in the Julia set

By using this sequence of annuli B_n, we can construct points in $J(f)$ with various "annular itineraries" including

- points in $A_R(f)$;
- points in $L(f)$;
- points that escape arbitrarily slowly;
- points that escape at least at a given rate and, in some sense, at no faster rate.

Covering lemmas

The key ingredients in the proof of the main theorem are two new covering lemmas of the following type.

Covering lemmas

The key ingredients in the proof of the main theorem are two new covering lemmas of the following type.

Lemma (Bergweiler, Rippon and Stallard, 2013)

There exist $K > 1$ and $R_0 > 0$ such that, if

$$f \text{ omits } 0 \text{ in } A(R, R'),$$

where $R'/R \geq K^2$ and $R \geq R_0$, then

Covering lemmas

The key ingredients in the proof of the main theorem are two new covering lemmas of the following type.

Lemma (Bergweiler, Rippon and Stallard, 2013)

There exist $K > 1$ and $R_0 > 0$ such that, if

$$f \text{ omits } 0 \text{ in } A(R, R'),$$

where $R'/R \geq K^2$ and $R \geq R_0$, then

$$f(A(R, R')) \supset A(M(R), M(R'/K)).$$

Covering lemmas

Lemma (Rippon and Stallard)

There exists $\epsilon(r) \to 0$ *as* $r \to \infty$ *such that, if*

$$\text{there exists } z_1 \in A(2r, 4r) \text{ with } |f(z_1)| < 1,$$

then

Covering lemmas

Lemma (Rippon and Stallard)

There exists $\epsilon(r) \to 0$ *as* $r \to \infty$ *such that, if*

$$\text{there exists } z_1 \in A(2r, 4r) \text{ with } |f(z_1)| < 1,$$

then

$$f(A(r, 8r)) \supset B(0, M(r))$$

Covering lemmas

Lemma (Rippon and Stallard)

There exists $\epsilon(r) \to 0$ *as* $r \to \infty$ *such that, if*

$$\text{there exists } z_1 \in A(2r, 4r) \text{ with } |f(z_1)| < 1,$$

then

$$f(A(r, 8r)) \supset B(0, M(r))$$

or

$$f(A(r, 8r)) \supset B(0, M(r)) \setminus B(w_1, \epsilon(r) \max\{|w_1|, 1\}),$$

for some $w_1 \in B(0, M(r))$.

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.
- Take z_2 with $|z_2| = s$ and $|f(z_2)| = M(s) \geq M(r)$.

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.
- Take z_2 with $|z_2| = s$ and $|f(z_2)| = M(s) \geq M(r)$.
- Suppose $f(A(s/2, 2s))$ omits w_1 and w_2, with w_2 not in specified disc.

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.
- Take z_2 with $|z_2| = s$ and $|f(z_2)| = M(s) \geq M(r)$.
- Suppose $f(A(s/2, 2s))$ omits w_1 and w_2, with w_2 not in specified disc.
- Then there exists absolute constant $C > 0$ such that

$$\begin{aligned} C &\geq \rho_{A(s/2,2s)}(z_1, z_2) \\ &\geq \rho_{f(A(s/2,2s))}(f(z_1), f(z_2)) \end{aligned}$$

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.
- Take z_2 with $|z_2| = s$ and $|f(z_2)| = M(s) \geq M(r)$.
- Suppose $f(A(s/2, 2s))$ omits w_1 and w_2, with w_2 not in specified disc.
- Then there exists absolute constant $C > 0$ such that

$$\begin{aligned} C &\geq \rho_{A(s/2,2s)}(z_1, z_2) \\ &\geq \rho_{f(A(s/2,2s))}(f(z_1), f(z_2)) \\ &\geq \rho_{\mathbb{C}\setminus\{w_1,w_2\}}(f(z_1), f(z_2)). \end{aligned}$$

Proof of second covering lemma

- Suppose $|z_1| = s$ and consider $A(s/2, 2s) \subset A(r, 8r)$.
- Take z_2 with $|z_2| = s$ and $|f(z_2)| = M(s) \geq M(r)$.
- Suppose $f(A(s/2, 2s))$ omits w_1 and w_2, with w_2 not in specified disc.
- Then there exists absolute constant $C > 0$ such that

$$\begin{aligned} C &\geq \rho_{A(s/2,2s)}(z_1, z_2) \\ &\geq \rho_{f(A(s/2,2s))}(f(z_1), f(z_2)) \\ &\geq \rho_{\mathbb{C}\setminus\{w_1,w_2\}}(f(z_1), f(z_2)). \end{aligned}$$

- This gives a contradiction – shown by applying a linear transformation mapping w_1 to 0 and w_2 to 1.