Regualrity and Irregularity of Fiber Dimensions for Non-Autonomous Systems

The Role of Complex Analysis in Complex Dynamics
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joint work with

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Take arbitrary

$$\lambda_1, \lambda_2, ..., \lambda_n, ... \in \Lambda_0, \text{ or, equivalently, } \lambda = (\lambda_1, \lambda_2, ...) \in \Lambda = \Lambda_0^{\mathbb{N}}.$$

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- Random dynamics: random choice of λ_n's.
- Non-autonomous dynamics: arbitrary choice of λ_n's.

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With the shift map

$$\sigma : \Lambda \to \Lambda, \quad \sigma(\lambda_1, \lambda_2, \lambda_3, ...) = (\lambda_2, \lambda_3, ...)$$

we have

$$\mathcal{J}_\lambda \xrightarrow{f_\lambda} \mathcal{J}_{\sigma(\lambda)} \xrightarrow{f_{\sigma(\lambda)}} \mathcal{J}_{\sigma^2(\lambda)} \xrightarrow{f_{\sigma^2(\lambda)}} \mathcal{J}_{\sigma^3(\lambda)} \quad ...$$

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Deterministic case:

- varying hyperbolic rational functions $\rightsquigarrow$ HD is real-analytic ([Ruelle]);
    - Julia–Lavourse maps ([M. U., M. Zinsmeister])
    - transcendental meromorphic functions ([M. U., A. Zdunik], [V. Mayer, M. U.])
    - semi-hyperbolic rational functions ([M. U.])
    - parabolic cubic polynomials ([H. Akter, M. U.])
- Random case (hyperbolic rational functions) : (H. H. Rugh).

For non-autononomous systems we have the following results.

Theorem (A)

Suppose Λ is equipped with the sup-topology $\|.\|_\infty$ and suppose that $\eta \in \Lambda$ is a "hyperbolic" parameter (in fact a stable parameter). Then

$$\lambda \mapsto \mathcal{J}_\lambda \text{ is continuous}$$

and

$$\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda) \quad \text{and} \quad \lambda \mapsto \mathrm{PD}(\mathcal{J}_\lambda)$$

are both Hölder continuous in some neighborhood of η with Hölder exponent $\alpha = \alpha(\lambda) \to 1$ if $\lambda \to \eta$.

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This is result is in some sense best possible:

Theorem (B)

Let

$$\Lambda_0 = \{\tau \in \mathbb{C} : |\tau| > 40\}.$$

Let $\Lambda = \Lambda_0^{\mathbb{N}}$ *be equipped with the sup-norm.*

Consider the quadratic family

$$\mathcal{F} = \left\{ f_\tau(z) = \frac{\tau}{2}(z^2 - 1) + 1 : \tau \in \Lambda_0 \right\}.$$

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$\Lambda = \Lambda^{hyp}$ *and no of the functions*

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Moreover, there exists an open and dense set $\Omega \subset \Lambda$ *such that*

$$\mathrm{HD}(\mathcal{J}_\lambda) < \mathrm{PD}(\mathcal{J}_\lambda) \quad \textit{for every} \quad \lambda \in \Omega\,.$$

The following example illustrates some pecularities of non-autonomous dynamics.

Example 1. Conjugate $f(z) = z^2$ by similarities $h_n(z) = \alpha_n z$. Then choose $\alpha_n > 0$ such that for the new maps f_{λ_n} we have

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- $h_1(\mathcal{J}(f)) \neq S^1 \cup \{\infty\} \rightsquigarrow$ conjugations do NOT in general preserve Julia sets.
- Initial (determinisitic) system is topologically exact whereas the new one is not.

Definition

f_λ is topologically exact if $\forall$ open set U with $U \cap \mathcal{J}_\lambda \neq \emptyset$ there exist $N \geq 1$ such that

$$f_\lambda^N(U) \supset \mathcal{J}_{\sigma^N(\lambda)}.$$

Definition

Two non-autonomous systems

$$f_\lambda, \ \ (\lambda = (\lambda_1, \lambda_2, \dots)) \quad \text{and} \quad f_\mu, \ \ (\mu = (\mu_1, \mu_2, \dots))$$

are called conjugated if there exist homeomorphisms

$$h_j : \mathcal{J}(f_{\lambda_j}) \to \mathcal{J}(f_{\mu_j})$$

such that

$$h_{j+1} \circ f_{\lambda_j} = f_{\mu_j} \circ h_j \quad \text{for every } \ j \geq 1 \, . \tag{1}$$

If in addition the families $\{h_j\}_j$ *and* $\{h_j^{-1}\}_j$ *are equicontinuous then*

f_λ *and* f_μ *are called bi-equicontinuous conjugated.*

If all homeomorphisms h_j *are (quasi)–conformal then we say that the systems* f_λ *and* f_μ *are (quasi)–conformally conjugated or (quasi)–conformally bi-equicontinuous conjugated.*

Stability

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A *holomorphic motion* of a set $E \subset \hat{\mathbb{C}}$ over V with basepoint η is a mapping $h : V \times E \to \hat{\mathbb{C}}$ having the following three properties.

- $h_\eta = id_E$,
- for every $\lambda \in \Lambda$, the map $z \mapsto h_\lambda(z)$ is injective on E and
- for every $z \in E$, $\lambda \mapsto h_\lambda(z)$ is a holomorphic map on Λ .

Case $V = \mathbb{D}$. [MSS]'s initial λ–Lemma states that the motion can be extended to $\overline{E}$. Bers-Royden, Slodkovsky, ...: Every holomorphic motion over $\mathbb{D}$ is the restriction of a motion of the whole $\hat{\mathbb{C}}$.

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Hubbard: the later is false if the parameter space V is higher-dimensional. However, we have:

Theorem (λ–Lemma (Mitra, Jiang-Mitra))

A holomorphic motion h of a set $E \subset \hat{\mathbb{C}}$ over a simply connected complex Banach manifold V with basepoint $\eta \in V$ extents to a holomorphic motion H of $\overline{E}$ over V such that

1. *for every $\lambda \in V$, the map H_λ is a global quasiconformal map of $\hat{\mathbb{C}}$ with dilatation bounded by $\exp(2\rho_V(\eta, \lambda))$ where ρ_V is the Kobayashi pseudometric on V.*
2. *the map $(\lambda, z) \mapsto H_\lambda(z)$ is continuous.*

Stability in the deterministic case: MSS, Lyubich.

$$\mathcal{F} = \{f_\lambda : \lambda \in \Lambda_0\} \quad (\text{for ex. } \mathcal{F} = \{z^2 + \lambda : \lambda \in \Lambda_0 = \mathbb{C}\}).$$

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- normal critical orbits $\lambda \mapsto c_\lambda \mapsto f_\lambda(c_\lambda) \mapsto f_\lambda^2(c_\lambda) \mapsto ... \mapsto f_\lambda^n(c_\lambda) \mapsto ...$

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- such that the following diagram commutes.

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\end{array}$$

Corollary

If $\eta \in \Lambda^{stable}$ then, in a NBH V of η,

$$\lambda \mapsto \mathcal{J}_\lambda \text{ is continuous} \quad \text{and} \quad \lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda) \text{ is Hölder continuous}$$

with Hölder exponents depending on λ.

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Then, the map f_η *is holomorphically stable if and only if*

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(1) $\alpha_i^n(\lambda) \in \mathcal{J}_{\sigma^n(\lambda)}$ *and* $\alpha_i^n(\lambda) \neq \alpha_j^n(\lambda)$ *for all $\lambda \in V$ and $i \neq j$.*

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(2) $f_\lambda^n\left(\mathcal{C}_{f_\lambda^n}\right) \cap \{\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda)\} = \emptyset$ *for all* $\lambda \in V$ *and* $n \geq 1$.

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(3) *if $\alpha_i^{n+k}(\lambda) = f_{\sigma^n(\lambda)}^k(\alpha_j^n(\lambda))$ for some $\lambda \in V$, then this equality holds for all $\lambda \in V$.*

(⇒) Consider the map f_η and define the points $\alpha_j^n(\eta)$ by induction. Since $\mathcal{J}_\eta$ is perfect, there exist three distinct points

$$\alpha_1^0(\eta), \alpha_2^0(\eta), \alpha_3^0(\eta) \in \mathcal{J}_\eta.$$

Suppose that all the points $\alpha_j^k(\eta)$ are defined for $0 \leq k < n$.

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The set $\mathcal{J}_{\sigma^n(\eta)}$ is also perfect and so there are distinct points

$$\alpha_1^n(\eta), \alpha_2^n(\eta), \alpha_3^n(\eta) \in \mathcal{J}_{\sigma^n(\eta)} \setminus \left[f_\eta^n\left(\mathcal{C}_{f_\eta^n}\right) \cup \bigcup_{k=0}^{n-1} f_{\sigma^k(\eta)}^{n-k}(\alpha_j^k(\eta)) \right].$$

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By our hypotheses there are holomorphic motions $\{h_{\sigma^n(\lambda)}\}_n$ such that stability holds. It suffices now to set

$$\alpha_j^n(\lambda) := h_{\sigma^n(\lambda)}(\alpha_j^n(\eta)) \quad \text{for every } \lambda \in V \text{ and all } n, j\,.$$

For the opposite direction we need the following:

Definition

f_η has normal critical orbits on the open set $V \subset \Lambda$ open if

- $\exists \kappa > 0$ *and*
- *three holomorphic functions $\alpha_i^n : V \to \hat{\mathbb{C}}$, $i = 1, 2, 3$, such that (2), (3) hold and*
- *(1')* $\operatorname{dist}_S(\alpha_i^n(\lambda), \alpha_j^n(\lambda)) \geq \kappa > 0 \quad \forall\ \lambda, n$ *and* $i \neq j$.

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The main technical result in here is the following theorem.

Theorem

Suppose that

$$\limsup_{n\to\infty} \mathrm{diam}\big(\mathcal{J}_{\sigma^n(\lambda)}\big) > 0\,.$$

If f_η has NCO, then f_η is holomorphically stable. Moreover, all involved holomorphic motions are bi-equicontinuous conjugations.

Proof. just for the first fiber $j = 0$.

Consider

$$E_{\lambda,n} := f_\lambda^{-n}\left(\{\alpha_1^n(\lambda), \alpha_2^n(\lambda), \alpha_3^n(\lambda), \}\right) \quad \textit{and} \quad \mathcal{E}_\lambda := \bigcup_{n\geq 0} E_{\lambda,n}\,.$$

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Since $dist_S(\alpha_i^n(\lambda), \alpha_j^n(\lambda)) \geq \kappa > 0$ for all n, λ and $i \neq j$, Montel's Theorem implies that $J_\lambda \subset \overline{\mathcal{E}}_\lambda$.
Thus, it suffices to construct suitable motions of $\mathcal{E}_\lambda$ and then to apply λ–Lemma.

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Thus, it suffices to construct suitable motions of $\mathcal{E}_\lambda$ and then to apply λ–Lemma.

$$z \in \mathcal{E}_\eta \quad \text{iff} \quad f_\eta^n(z) = \alpha_i^n(\eta) \ \text{ for some } i, n.$$

Define a holomorphic map $\lambda \mapsto z_\lambda$ using conditions (2) and (3), by applying the Implicite Function Theorem applied to the equation

$$f_\lambda^n(z_\lambda) = \alpha_i^n(\lambda)\,.$$

... □

The concluding argument is that we can conjugate our original system via Möbius transformations (not necessarily equicontinuous) to a system with fixed (0, 1, ∞) points $\alpha_j^n(\lambda)$, so having normal critical orbits. Then apply the above theorem (with NCO).

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Theorem

Suppose that

- *$\mathcal{F}$ contains at least two deterministic hyperbolic maps having Julia sets with different Hausdorff dimensions.*

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Then the function $\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda)$ is discontinuous at every point of Λ.

How big is Λ^{stable}?

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Proof.

Let $\eta \in \Lambda$. By our hypothesis there exists $f_{\lambda_0} \in \mathcal{F}$, a deterministic hyperbolic map, such that

$$\mathrm{HD}(\mathcal{J}(f_{\lambda_0})) \neq \mathrm{HD}(\mathcal{J}(f_\eta)).$$

Consider then the sequence

$$\lambda^{(n)} = (\eta_1, \eta_2, ..., \eta_n, \lambda_0, \lambda_0, \lambda_0, ...) \, .$$

On the one hand we have that

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for every $n \geq 1$, and hence

$$\mathrm{HD}(\mathcal{J}_{\lambda^{(n)}}) \not\to \mathrm{HD}(\mathcal{J}_\eta) \ \text{ as } \ n \to \infty.$$

□

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Proposition

With the Tychonov topology, $\Lambda^{stable} = \emptyset$.

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$$|(f^n_{\sigma^j(\lambda)})'(z)| \geq c\gamma^n \quad \text{for all} \quad z \in \mathcal{J}_{\sigma^j(\lambda)} \quad \text{and all } n \geq 1 \ .$$

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- u-hyperbolic maps are topologically exact.
- There is also a topological description of such hyperbolic maps.

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5. *the critical points of $f_{\sigma^j(\lambda)}$ are all contained in U_j.*

Corollary

Uniform hyperbolicity is an open condition for the l^∞-topology on Λ (but not for the Tychonov topology).

Proposition

If f_η is a uniform hyperbolic map, then f_η has normal critical orbits on some open neighborhood $V \subset \Lambda$ of η.

So,

Corollary

$\Lambda^{uHyp} \subset \Lambda^{stable}$ *if Λ is endowed with the l^∞- topology.*

Thus,

Theorem

The the functions

$$\Lambda^{uHyp} \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda) \quad \textit{and} \quad \Lambda^{uHyp} \ni \lambda \mapsto \mathrm{PD}(\mathcal{J}_\lambda)$$

(in fact all fractal dimensions) are Hölder continuous with Hölder exponent $\alpha(\lambda) \to 1$ if λ converges to the some point $\eta \in \Lambda^{uHyp}$.

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For every $t \geq 0$:

- (Transfer) operators $\mathcal{L}_{j,t} : \mathcal{C}(\mathcal{J}_j) \to \mathcal{C}(\mathcal{J}_{j+1})$ are defined by

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$$\mathcal{L}_{j,t}^*(m_{j+1,t}) = \rho_{j,t} m_{j,t} \quad \text{for all} \ \ j \geq 0 \, .$$

- Although conformal measures are not unique, however

$$\frac{1}{B_t} \leq \tilde{\rho}_{j,t}/\rho_{j,t} \leq B_t .$$

- The zooming procedure and Besicovitch covering arguments give

$$\frac{m_{j,t}(B(z,r))}{r^t} \simeq \rho_{j,t}^{-n},$$

  where

$$\rho_{j,t}^n = \rho_{j,t}\rho_{j+1,t} \dots \rho_{j+n-1,t}$$

  and $n \geq 1$ is maximal s.t. $|(f_j^n)'(z)|^{-1} \geq r/\delta$.

Lower pressure

$$\underline{\mathrm{P}}_\lambda(t) := \liminf_{n\to\infty} \frac{1}{n} \log \rho^n_{\lambda,t} = \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_{\lambda,t} 1\!\!1(w_n),$$

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Proposition

Both upper and lower pressures are continuous and strictly decreasing. Precisely, there are $A, B > 1$ such that if $0 \le t_1 < t_2$, then

$$-(t_2 - t_1)\log A \le \underline{\mathrm{P}}_\lambda(t_2) - \underline{\mathrm{P}}_\lambda(t_1) \le -(t_2 - t_1)\log B \tag{2}$$

and the same relation is true for the upper pressure $\overline{\mathrm{P}}_\lambda$.

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$$\mathcal{F} = \left\{ f_\tau(z) = \frac{\tau}{2}(z^2 - 1) + 1 : \tau \in \Lambda_0 \right\}, \quad \text{where} \quad \Lambda_0 = \{|\tau| > 40\}.$$

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Proposition

For every $\eta \in \Lambda$ and every $t > 0$ there exists a sequence $s = (s_0, s_1 \dots) \in \{-1, 1\}^{\mathbb{N}}$ such that for all $x \in (-r, r)$ with sufficiently small $r > 0$:

$$\overline{\mathrm{P}}_{\lambda(x)}(t) \geq \overline{\mathrm{P}}_\eta(t) + \frac{t}{2}|x| \quad \text{and} \quad \underline{\mathrm{P}}_{\lambda(x)}(t) \leq \underline{\mathrm{P}}_\eta(t) - \frac{t}{2}|x|.$$

In particular, no of the functions $\lambda \mapsto \underline{\mathrm{P}}_\lambda(t)$ and $\lambda \mapsto \overline{\mathrm{P}}_\lambda(t)$ is differentiable at any point $\eta \in \Lambda$.

Proof of Theorem 1.2.

Hausdorff dimension:

$$\underline{\mathrm{P}}_{\lambda(x)}(\underline{h}_\eta) \leq \underline{\mathrm{P}}_\eta(\underline{h}_\eta) - \frac{\underline{h}_\eta}{2}|x| = -\frac{\underline{h}_\eta}{2}|x| < 0\,.$$

Since the pressures is strictly decreasing, this gives

$$\underline{h}_{\lambda(x)} < \underline{h}_\eta.$$

Therefore, Proposition 5.1 yields

$$0 = \underline{\mathrm{P}}_{\lambda(x)}(\underline{h}_{\lambda(x)}) \leq \mathrm{P}_{\lambda(x)}(\underline{h}_\eta) + (\underline{h}_\eta - \underline{h}_{\lambda(x)}) \log A \leq -\frac{\underline{h}_\eta}{2}|x| + (\underline{h}_\eta - \underline{h}_{\lambda(x)}) \log A$$

from which it follows that

$$\underline{h}_{\lambda(x)} \leq \underline{h}_\eta\left(1 - \frac{|x|}{2\log A}\right)\,. \tag{3}$$

Therefore, the function

$$x \mapsto \underline{h}_{\lambda(x)} = \mathrm{HD}(\mathcal{J}_{\lambda(x)})$$

is not differentiable at 0. □

Similarly to (3) one obtains,

$$\overline{h}_{\lambda(x)} \geq \overline{h}_{\eta}\left(1 + \frac{|x|}{2\log A}\right) \tag{4}$$

and non-differentiability of the packing dimension follows.

In any family $\mathcal{F}$ the set

$$\Omega = \{\lambda \in \Lambda : \mathrm{HD}(\mathcal{J}_\lambda) < \mathrm{PD}(\mathcal{J}_\lambda)\}$$

is open in $l^\infty(\Lambda)$ because of continuity of both dimensions.

The density of Ω for our particular quadratic family considered in this section follows immediately from (3) and (4) and the inequality $\underline{h}_\eta \leq \overline{h}_\eta$.

THANK YOU