On the measure of the escaping set of a quasiregular analogue of sine

Sebastian Vogel

Christian-Albrechts-Universität zu Kiel

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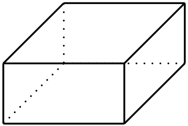
We want to generalise this theorem for a quasiregular analogue of sine.

$\mathbb{R}^d$



$[-1,1]^{d-1} \times [0,1]$

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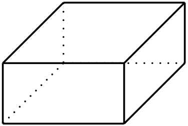
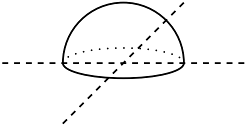


$[-1,1]^{d-1} \times [0,1]$

h

bi-Lipschitz



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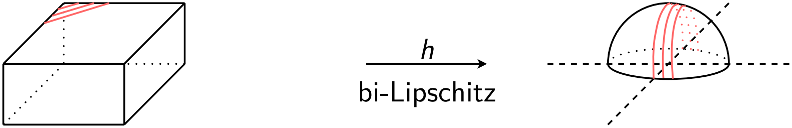

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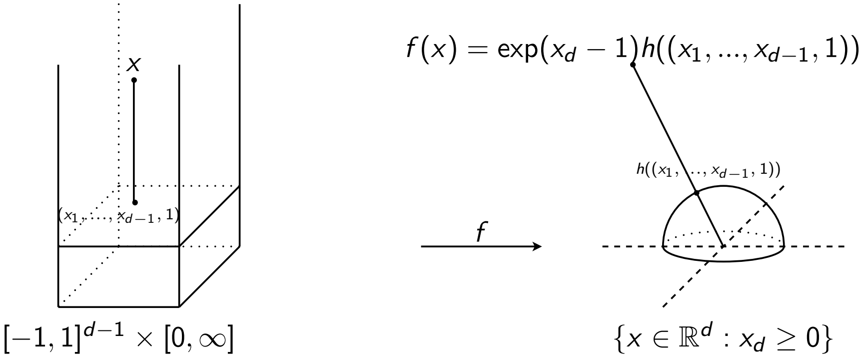
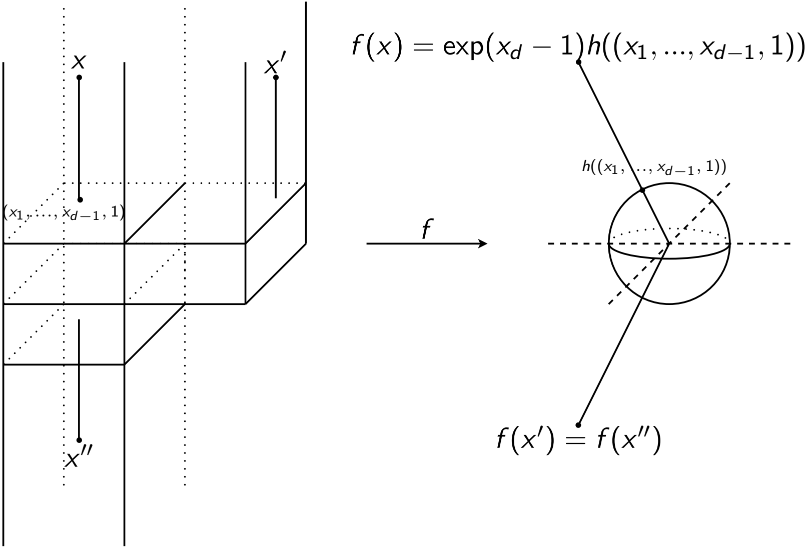

x
(x_1, ..., x_{d−1}, 1)
[−1, 1]^{d−1} × [0, ∞]
f(x) = exp(x_d − 1)h((x_1, ..., x_{d−1}, 1))
h((x_1, ..., x_{d−1}, 1))
f
{x ∈ ℝ^d : x_d ≥ 0}


x
x'
$(x_1, \ldots, x_{d-1}, 1)$
x''
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- f is differentiable almost everywhere.

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Theorem (Schleicher 2007)

There exists a representation of $\mathbb{C}$ *as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.*

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For λ sufficiently large, the periodic points of $\widetilde{f} = \lambda f$ are dense in $\mathbb{R}^d$ (and all repelling).
Furthermore $\widetilde{f}$ has the blowing-up property everywhere in $\mathbb{R}^d$, that is

$$\bigcup_{k=0}^{\infty} \widetilde{f}^k(U) = \mathbb{R}^d, \quad \text{for any non-empty open set } U \subset \mathbb{R}^d.$$

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Theorem

Let f be the quasiregular analogue of sine. Then

$$\text{meas}(I(f)) > 0,$$

where meas *denotes the d-dimensional Lebesgue measure.*

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Then $L = T_0$ and we put

$$S := \mathbb{R}^d \setminus L.$$

Definition

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For $x \in \mathbb{R}^d$ we denote by

$$Q(x) := \left\{ y \in \mathbb{R}^d : |y_j - x_j| \leq \frac{|x_d|}{2} \right\}$$

the axis parallel cube around x with edges of length $\frac{|x_d|}{2}$.

Lemma

For x_0 large and $|x_d| \geq x_0$ we have

$$\operatorname{dens}(S, Q(x)) \leq 2\widetilde{L}^4 \exp\left(-\frac{|x_d|}{4} + \frac{1}{2}\right) =: 2\widetilde{L}^4 \delta(|x_d|),$$

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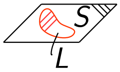

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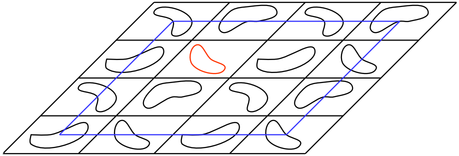

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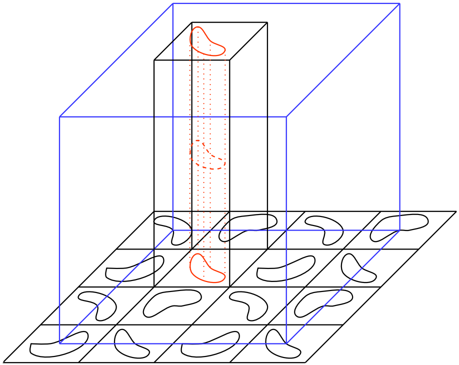
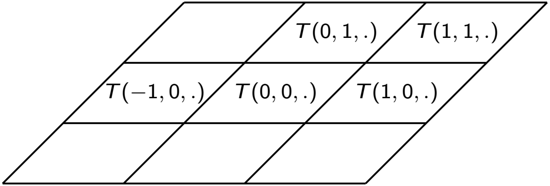


We put $R := \mathbb{Z}^{d-1} \times \{-1, 1\}$ and for $r = (r_1, ..., r_d) \in R$ we define

$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j| \leq 1 \text{ for } 1 \leq j \leq d-1,\ r_d x_d \geq 0\}$$



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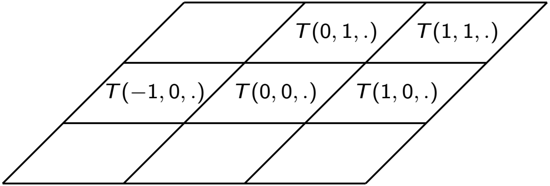

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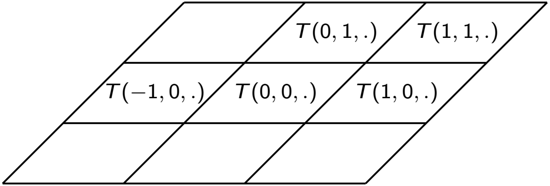

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For $r \in R$ we denote by Λ^r the inverse function of $f|_{T(r)}$, thus $\Lambda^r : \mathbb{H}^+ \to T(r)$ or $\Lambda^r : \mathbb{H}^- \to T(r)$ depending on r.

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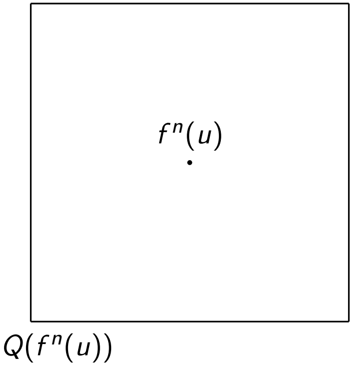

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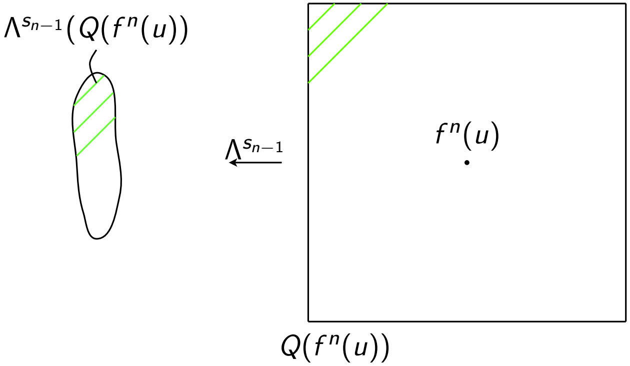

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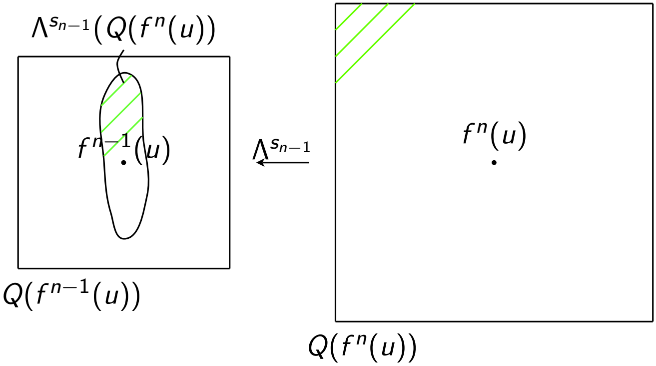

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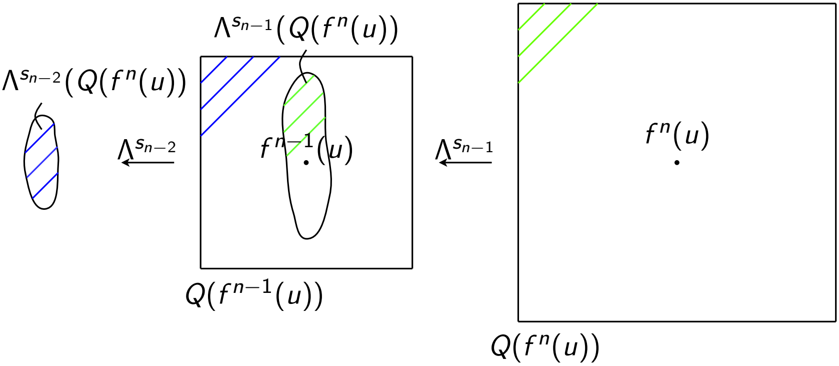

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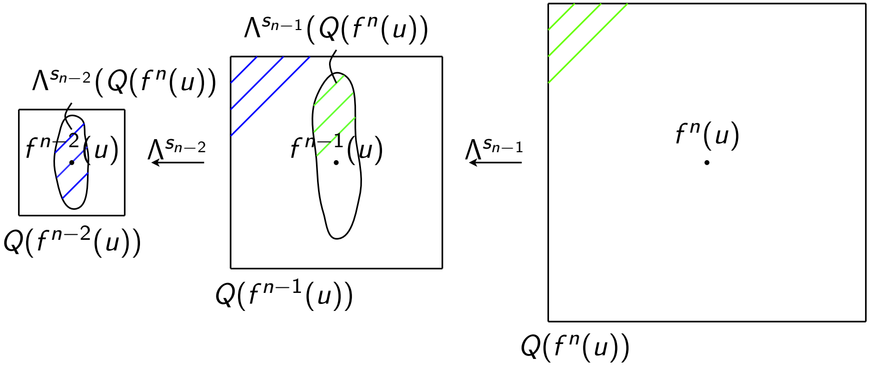

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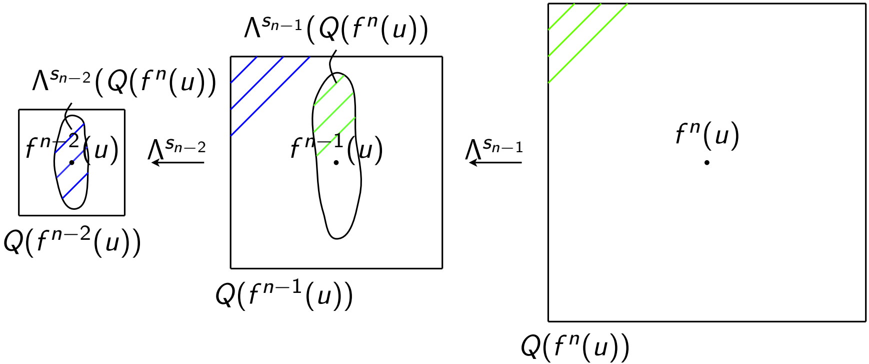

So we have $\Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \subset Q(f^{n-j-1}(u))$ for x_0 large and $0 \leq j \leq n-1$.

We denote by $B(u, r_n(u)) = B(u, r_n)$ the largest ball around u in $\varphi_n(Q(f^n(u)))$.

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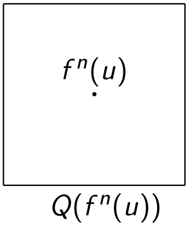

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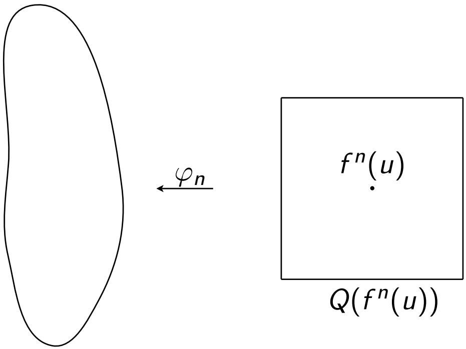

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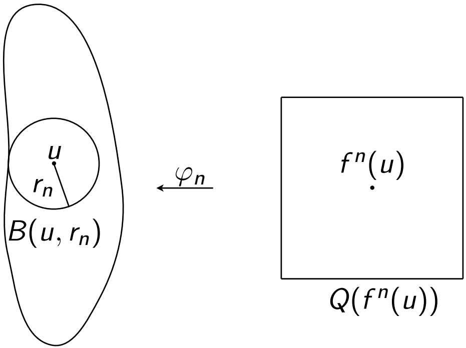
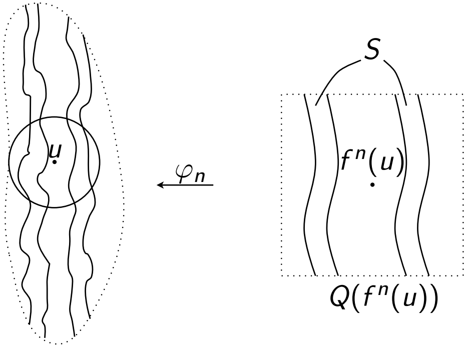


Lemma

For x_0 large we have

$$\mathrm{dens}(f^{-n}(S), B(u, r_n)) \leq \tilde{\eta}\delta\left(\mathrm{E}_{\frac{1}{2}}^{n}(x_0)\right) K^n =: \tilde{\eta}\delta(x_{0,n})K^n$$

for some constants $\tilde{\eta}, K \geq 1$*, where* $E_{\frac{1}{2}} : \mathbb{R} \to \mathbb{R},\ x \mapsto \exp\left(\frac{1}{2}x\right)$



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&\leq \frac{2^d}{c_d} \delta \left(\mathrm{E}^n_{\frac{1}{2}}(x_0) \right) \prod_{j=0}^{n-1} \frac{\sup\limits_{y \in Q(f^{n-j}(u))} |J_{\Lambda^{s_{n-j-1}}}(y)|}{\left(\inf\limits_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d}
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&\leq \frac{2^d}{c_d} \delta \left(\mathrm{E}^n_{\frac{1}{2}}(x_0) \right) \left(\frac{c_3}{c_1^d} \left(1 + 2\sqrt{d} \right)^d \right)^n
\end{aligned}
$$

Theorem (Besicovitch covering lemma)

Let $M \subset \mathbb{R}^d$ be bounded, $r : M \to]0, \infty[$. Then there exists an at most countable subset A of M satisfying

$$M \subset \bigcup_{x \in A} B(x, r(x))$$

such that no point in $\mathbb{R}^d$ is contained in more than 4^{2d} of the balls $B(x, r(x))$, $x \in A$.

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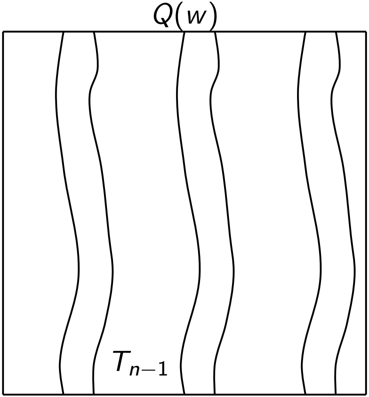

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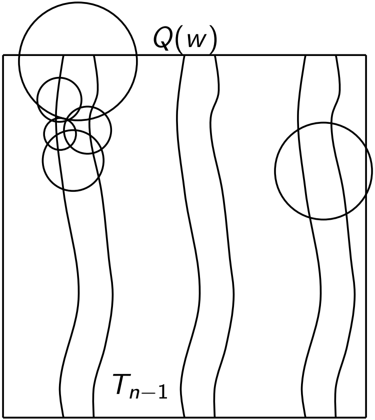

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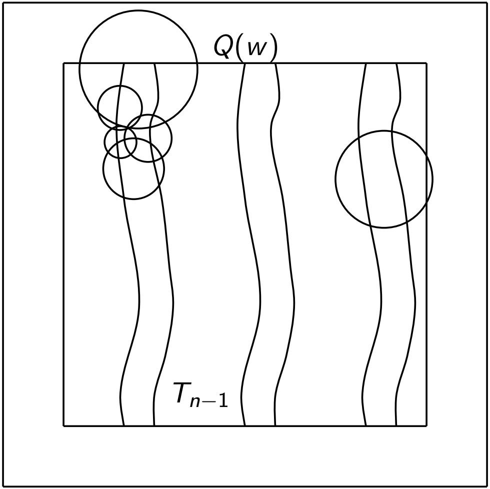

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$$\text{dens}(f^{-n}(S), T_{n-1} \cap Q(w)) = \text{dens}(T_{n-1} \setminus T_n, T_{n-1} \cap Q(w)) \leq \eta\delta(x_{0,n})K^n$$

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Hence we have

$$\text{dens}(T, T_0 \cap Q(w)) > 0$$

and $\text{meas}(T) > 0$ and since $T \subset I(f)$ we get $\text{meas}(I(f)) > 0$.

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In the case of the quasiregular analogue of sine we have

Theorem

Let $T(r)$ be a tract of f. Then $T(r) \setminus I(f)$ has finite measure.

Thank you very much for your attention.