Hausdorff and harmonic measures on non-homogeneous Cantor sets

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Plan

1. Some facts about harmonic measure in the plane
2. Old and new results
3. Outline of proofs
4. Further comments and remarks

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Harmonic measure- definition

If Ω is a simply -connected domain and $R : D \to \Omega$ is the Riemann map, $R(0) = a$ then ω_a can be defined as

$$\omega_a = R_*(Leb)$$

(angular limit f R exists a.e)
For an arbitrary domain $\Omega \subset \overline{\mathbb{C}}$ such that $cap(\Omega \setminus \overline{\mathbb{C}}) > 0$ the harmonic measure can be defined by the Wiener's solution to the Dirichlet problem: If $f \in C(\partial\Omega)$ then

$$\int_{\partial\Omega} f(w) d\omega(z, w, \Omega) = u_f(z)$$

where $u_f(z)$ is the Wiener's solution to the Dirichlet problem for f.
General question: How is ω distributed over the boundary of Ω?

Dimension of measure

We deal with the Hausdorff dimension and the harmonic measure of a certain type of Cantor sets X in the plane.

Dimension of measure

Recall the definition of the Hausdorff dimension of a (probability) Borel measure μ:

$$\dim_H(\mu) = \inf_{Z:\mu(Z)=1} \dim_H(Z)$$

where infimum is taken over all Borel subsets Z with $\mu(Z) = 1$.

Dimension of harmonic measure

Let ω be the harmonic measure on $\hat{\mathbb{C}} \setminus X$ evaluated at ∞. By celebrated results of N. Makarov and P. Jones, T. Wolff the Hausdorff dimension of ω is not larger than one.

Some previous results

Self-conformal sets in the plane

It has been observed, for several self-similar, self-conformal sets, or, more generally, conformal repellers, that $\dim_H(\omega) < \dim_H(X)$ (Batakis, Volberg, Makarov and Volberg, Zdunik, Urbański and Zdunik). Nevertheless, the intriguing question about the inequality of dimensions for an arbitrary self -conformal Cantor repeller, remains open.

On the other hand, the inequality $\dim_H(\omega) < \dim_H(X)$ is not true for more general Cantor sets, even after assuming a strict regularity of the construction (Batakis).

Harmonic measure in $\mathbb{R}^d$

In $\mathbb{R}^d$, $d \geq 3$, a general result of Bourgain states that for all domains Ω, the dimension of harmonic measure is bounded above by $d - \epsilon(d)$, where $\epsilon(d)$ is a positive constant depending only on d, whose exact value remains unknown.

Methods

All the proofs of the strict inequality $\dim_H(\omega) < \dim_H(X)$ for conformal repellers rely on the ergodic theory tools: one constructs an invariant measure equivalent to the harmonic measure and its ergodic properties play a crucial role in the arguments.

We prove the inequality $\dim_H(\omega) < \dim_H(X)$ for a class of non-homogeneous Cantor sets. In this case there is no invariant ergodic measure equivalent to harmonic measure and hence previously mentioned tools are inapplicable.

Admissible family and admissible maps

Let Q be a Jordan domain in $\mathbb{C}$. Let $M > 0$, $0 < \underline{a} < \overline{a} < 1$ be fixed. We fix a positive integer $N > 1$. Let $\mathcal{Q} = (Q_1, \ldots Q_N)$ be a family of Jordan domains such that each Q_i is a preimage of Q under some (expanding) similitude $(a_i)^{-1}z + b_i$.

We call a family $\mathcal{Q} = (Q_1, \ldots Q_N)$ *admissible* if the following holds:

1. $\underline{a} \leq |a_i| \leq \overline{a}$
2. $\mathrm{clQ_i} \subset \mathrm{Q}$
3. there exists an annulus $A \subset Q$ with $mod(A) > M$ and separating ∂Q from $\bigcup_j Q_j$ (i.e ∂Q and $\bigcup_j Q_j$ are in different components of $\mathbb{C} \setminus A$.

In this way, we have introduced a piecewise linear map f defined on the union of admissible discs: $f : \bigcup_{Q_i \in \mathcal{Q}} Q_i \to Q$ by the formula

$$f(z) = \sum_{i=1}^{N} (a_i^{-1} z + b_i) 1_{Q_i},$$

where $a_i^{-1} Q_i + b_i = Q$. If $\mathcal{Q}$ satisfies the conditions $(1-3)$, then we call the map f admissible.

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Admissible Cantor set

Definition

A set $X_0 \subset \mathbb{C}$ is called admissible if

$$X_0 = \bigcap_{n=1}^{\infty} (f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0)^{-1} (Q).$$

for some sequence of admissible maps f_k:

$$f_k(z) = \sum_{i=1}^{N} (a_{k,i}^{-1} z + b_{k,i}) 1_{Q_{k,i}},$$

where $a_{k,i}^{-1} Q_{k,i} + b_{k,i} = Q$. So, the map f_k is defined on the union of the domains $\{Q_{k,i}\}_{i=1}^{N}$, and $f_k\,(Q_{k,i}) = Q$, for all $i = 1, \ldots, N$.

It follows from the construction that X_0 is a Cantor set.

Results

Theorem A

Let X be an admissible Cantor set. Let ω be the harmonic measure on X. Then

$$\dim_H(\omega) < \dim_H(X).$$

Theorem B

Let $(f_k)(z) = \sum_{i=1}^{N}(a_{k,i}^{-1}z + b_{k,i})1_{Q_{k,i}}$ be a sequence of admissible maps and let $X = X_0$ be the associated admissible Cantor set. There exist a sequence of admissible functions $(\tilde{f}_k)$, such that

1. $\lim_{k\to\infty} \max_i(|\tilde{a}_{k,i} - a_{k,i}| + |b_{k,i} - \tilde{b}_{k,i}|) = 0$
2. the associated Cantor set $\tilde{X}$ is admissible and $\dim_{\mathcal{H}}(\tilde{X}) = \dim_{\mathcal{H}}(X)$
3. $0 < H_{\dim_{\mathcal{H}}(\tilde{X})}(\tilde{X}) < \infty$.
4. If ω and $\tilde{\omega}$ are the harmonic measures of X and $\tilde{X}$ respectively, then $\dim \omega = \dim \tilde{\omega}$.

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Hausdorff dimension

The following simple proposition gives an explicit formula for the Hausdorff dimension of the set X_0.

Let $|a_{k,1}|, \dots |a_{k,N}|$ be the sequence of 'scales" used in the construction of X_0. Then $\rho = dim_H(X_0)$ is characterized in the following way:

$$\rho = \inf\{s : \liminf_{n\to\infty} \prod_{k=1}^{n} (|a_{k,1}|^s + |a_{k,2}|^s + \dots |a_{k,N}|^s) = 0\} \tag{1}$$

Conformal measure

Let, as above, X_0 be an admissible set, $X_k = f^k(X_0)$.

Fix $h > 0$. The sequence of probability measures $\nu_0, \nu_1, \ldots$ is called a collection of $h-$ conformal measures if $\mathrm{supp}\nu_k = X_k$ and the following holds: there exists a sequence $\lambda_{k,h}$ of positive "scaling factors" such that for every Borel measurable set $B \subset Q_{k,i}$:

$$\nu_{k+1}(f_k(B)) = \lambda_{k,h} \cdot (|a_{k,i}|^{-h}) \cdot \nu_k(B) = \lambda_{k,h} \int_B |f_k'|^h d\nu_k$$

The collection of h- conformal measures exists and it is unique. In particular, if ρ is the common value of Hausdorff dimension of the sets X_k and the ρ-dimensional Hausdorff measure H_ρ of X_0 (and thus of all X_k) is positive and finite then the collection of normalized Hausdorff measures can be taken as ρ- conformal measures ν_k. The normalizing factor are given explicitly: $\lambda_{k,h} = (|a_{k,1}|^h + \ldots |a_{k,N}|^h)$ for all k.

Proposition

Let $X = X_0$ be an admissible Cantor set. Let, as above, $\omega = \omega_0$ be the harmonic measure on X_0, $\rho = dim_H(X)$ and let $\nu = \nu_0$ be the ρ-conformal measure on X_0. Assume the following:

* There exists $K > 0$ and $\gamma > 1$ such that for every cylinder $I = (I)_n \subset X$ of length n there exists a subcylinder $IJ = (IJ)_{n+K(I)}$, $K(I) \leq K$ such that

$$\max\left(\frac{\omega(IJ)}{\omega(I)} : \frac{\nu(IJ)}{\nu(I)}, \frac{\nu(IJ)}{\nu(I)} : \frac{\omega(IJ)}{\omega(I)}\right) > \gamma.$$

Then $\dim_H(\omega) < \dim_H(X) - \delta$ where δ is a constant depending only on $\underline{a}$, K, N, γ.

The alternative case

Proposition

Suppose that for all $1 > \gamma > 0$ and $K \in N$ there exist a cylinder I such that for all subcylinders IJ, where J is a word of length $\leq K$ we have

$$\gamma < \left| \frac{\omega(IJ)}{\omega(I)} : \frac{\nu_0(IJ)}{\nu_0(I)} \right| < \frac{1}{\gamma}. \tag{2}$$

Then we can construct another admissible Cantor set $\tilde{X}$ (not necessarily of dimension ρ), a ρ-conformal measure $\tilde{\nu}$ on $\tilde{X}$ and a bounded subharmonic function $F \in \mathcal{G}_{\tilde{X}}$ such that $\Delta F = \tilde{\nu}$.

$\mathcal{G}_X$ denotes the family of all functions $F : Q \to \mathbb{R}$ such that F is continuous in Q, $F_{|Q \setminus X}$ is harmonic and strictly positive, while $F_{|X} = 0$.

Final conclusion

Finally, we prove the following result which implies that the "alternative case" considered above cannot hold. We use some ideas due to A. Volberg.

Let $X = X_0$ be an admissible Cantor set, and let $(\nu_k)_{k=0}^{\infty}$ be the collection of associated ρ conformal measures, where ρ is not necessarily equal to the Hausdorff dimension of the sets X_k. Further, let $\tilde{G} \in \mathcal{G}_X$ and let $\tilde{\omega} = \Delta \tilde{G}$. Then the measures $\tilde{\omega}$ and $\nu = \nu_0$ do not coincide.

1. Modulo some technical but small modifications the proofs can be carried out if we consider sequences of admissible funtions (f_n) with varying multiplicities $2 \leq N_n \leq N$.
2. Within this family of admissible Cantor sets, our proof gives a a uniform positive lower bound on $\dim X - \dim \omega$.
3. However, we have the following:

Proposition

There exists an (unbounded) sequence N_n and a sequence of admissible functions (f_n) of multiplicities N_n such that the dimension of harmonic measure ω of the Cantor set X associated to (f_n) is equal to the Hausdorff dimension of the set.

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