HYPERBOLIC METRIC AND WANDERING DOMAINS OF MEROMORPHIC FUNCTIONS

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1. Notations

Ω—a hyperbolic open set in the complex plane;

$\lambda_\Omega(z)$—the hyperbolic density on Ω;

$d_\Omega(u,v)$— the hyperbolic distance of u and v on Ω ($d_\Omega(u,v)=\infty$ for u and v in distinct component of Ω).

The hyperbolic density of the annulus $A=\{z\in\mathbf{C}: r<|z|<R\}$

$$\lambda_A(z)=\frac{\pi}{2\,|z|\operatorname{mod}(A)\sin(\pi\log(R/\,|z|)/\operatorname{mod}(A))},\forall\ z\in A, \tag{1.1}$$

where $\operatorname{mod}(A)=\log R/r$ is the module of A and on $|z|=\sqrt{Rr}$, $\lambda_A(z)=\frac{\pi}{2|z|\operatorname{mod}(A)}$.

The hyperbolic density of $\mathbf{C}\backslash\{a,b\}$ with $a\neq b$ in $\mathbb{C}$

$$\lambda_{a,b}(z)\geq\left[2\left|\frac{z-a}{b-a}\right|\left(\left|\log\left|\frac{z-a}{b-a}\right|\right|+\kappa\right)\right]^{-1},$$

where $\kappa=\Gamma(\frac{1}{4})^4/(4\pi^2)=4.3768796....$

2. Domain Constants

Set

$$C_\Omega = \inf\{\lambda_\Omega(z)\delta_\Omega(z) : \ \forall\, z \in \Omega\},$$

where $\delta_\Omega(z) = \inf\{|z-a| : \ \forall\, a \in \partial\Omega\}$.

For a simply connected domain Ω,

$$\frac{1}{4} \leq \mathcal{C}_\Omega \leq \frac{1}{2}.$$

And Ω is convex if and only if $C_\Omega = \frac{1}{2}$.

In fact, I think that for any $\frac{1}{4} \leq c \leq \frac{1}{2}$ there exists a simply connected domain Ω such that $\mathcal{C}_\Omega = c$.

A hyperbolic open set Ω is called to be uniformly perfect provided that $\mathcal{C}_\Omega > 0$ after Beardon and Pommerenke.

Definition 1. *A hyperbolic domain is called BP domain after A. F. Beardon and Ch. Pommerenke if it is uniformly perfect; An universal covering mapping p from the unit disk $\mathbb{D}$ onto a BP domain is called the BP function on $\mathbb{D}$, that is to say, $C_{p(\mathbb{D})} > 0$.*

A BP domain and BP function are similar to a simply connected domain and a univalent function in the sense of the following

Theorem A (Zheng). *Let $f(z)$ be a BP function on $\mathbb{D}$. Then*

$$\frac{d}{2}|f'(0)|\frac{(1-|z|)^{1/d-1}}{(1+|z|)^{1/d+1}} \le |f'(z)| \le \frac{2}{d}|f'(0)|\frac{(1+|z|)^{1/d-1}}{(1-|z|)^{1/d+1}}, \tag{2.1}$$

$$|\arg f'(z) - \arg f'(0)| \le \frac{2}{d}\log\left|\frac{1+|z|}{1-|z|}\right|$$

and

$$|f(z)-f(0)| \le |f'(0)|\left[\left(\frac{1+|z|}{1-|z|}\right)^{1/d} - 1\right],$$

where $d = 2C_{f(\mathbb{D})} > 0$. And

$$\{z : |z - f(0)| < C_{f(\mathbb{D})}|f'(0)|\} \subset f(\mathbb{D}).$$

When $f(\mathbb{D})$ is simply connected, then f is univalent on $\mathbb{D}$. From (2.1) we can write

$$\frac{1}{4}|f'(0)|\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le 4|f'(0)|\frac{1+|z|}{(1-|z|)^3}. \tag{2.2}$$

Comparing it to the Koebe distortion theorem, we guess that the coefficients $\frac{d}{2}$ and $\frac{2}{d}$ should be removed in (2.1).

Note that for a simply connected domain $f(\mathbb{D})$, $\frac{1}{2} \le d \le 1$ and $0 \le \frac{1}{d} - 1 \le 1$. So $(1-|z|)^{\frac{1}{d}-1} \ge 1 - |z|$, from this point of view, (2.1) is more precise than the Koebe's.

If $f(\mathbb{D})$ is convex, from (2.1) we have

$$\frac{1}{2}|f'(0)|\frac{1}{(1+|z|)^2} \le |f'(z)| \le 2|f'(0)|\frac{1}{(1-|z|)^2}.$$

Problem 1. *Could the coefficients $d/2$ and $2/d$ in (2.1) be removed?*

Equivalently, for a BP function $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, should we have

$$|a_2| \le \frac{1}{2C_{f(\Delta)}}?$$

$\mathcal{A}_\Omega$—-the set of non-trivial doubly connected domains A in Ω.

$$\mathrm{Mod}(\Omega) = \sup_{A\in\mathcal{A}_\Omega}\{\mathrm{mod}(A)\}.$$

In order to find relation between $\mathrm{Mod}(\Omega)$ and C_Ω, Beardon and Pommonenko in [5] introduced the notation

$$\beta_\Omega(z) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right| :\ a, b \in \partial\Omega\right\},\ z \in \Omega.$$

In view of the $\beta_\Omega(z)$ it is proved that $C_\Omega > 0$ if and only if $\mathrm{Mod}(\Omega) < \infty$.

For $a \notin \Omega$ we define

$$C_\Omega(a) = \inf\{\lambda_\Omega(z)|z-a| :\ z \in \Omega\},$$

$\mathrm{Mod}_a(\Omega) = \sup\{\mathrm{mod}(A) :\ A$ is a doubly connected domain in Ω and goes around $a\}$

and for $z \in \Omega$

$$\beta_\Omega(z; a) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right| :\ b \in \partial\Omega\right\},$$

If $\beta_\Omega(z_0; a) > 0$ for a $z_0 \in \Omega$, then

$$\Omega \supset \{z; e^{-\beta_\Omega(z_0;a)}|z_0 - a| < |z - a| < e^{\beta_\Omega(z_0;a)}|z_0 - a|\}.$$

Lemma 1. *(Beardon and Pommerenko, Zheng) We have*

$$\frac{1}{\beta_\Omega(z;a)+\kappa} \le \lambda_\Omega(z)\,|z-a| \le \frac{1}{2\beta_\Omega(z;a)} \tag{2.3}$$

for $z \in \Omega$ and $a \notin \Omega$.

By $A(r, R)$ we denote the annulus $\{z : \ r < |z| < R\}$ below.

Theorem B (Bergweiler, Rippon and Stallard) *There exists a $\delta > 0$ such that for any analytic function f on U with $0 \notin f(U)$, if there exist two points $z_1, z_2 \in U$ such that*

$$\rho_U(z_1, z_2) < \delta \text{ and } |f(z_1)| \geq 2|f(z_2)|,$$

then we have

$$f(U) \supset A(|f(z_2)|, |f(z_1)|). \tag{2.4}$$

In particular, for $K > \exp(\frac{\pi}{2}\delta)$, if f is analytic on $U = \{z : \ K^{-1}R < |z| < KR\}$ with $0 \notin f(U)$ and $|f(z_1)| \geq 2|f(z_2)|$ for $|z_1| = |z_2| = R$, then the inclusion (2.4) holds.

Theorem 1. (Zheng) *Let $f(z)$ be analytic on U with $0 \notin f(U)$. If there exist two distinct points z_1 and z_2 in U such that $|f(z_1)| > e^{\kappa\delta}|f(z_2)|$, where $\delta = d_U(z_1, z_2)$, then there exists a point $\hat{z} \in U$ such that $|f(z_2)| \leq |f(\hat{z})| \leq |f(z_1)|$ and*

$$f(U) \supset A\left(e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{1/\delta}|f(\hat{z})|,\ e^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}|f(\hat{z})|\right); \tag{2.5}$$

If $|f(z_1)| \geq \exp\left(\frac{\kappa\delta}{1-\delta}\right)|f(z_2)|$ and $0 < \delta < 1$, then

$$f(U) \supset A(|f(z_2)|, |f(z_1)|). \tag{2.6}$$

In particular, for $\delta \leq \frac{1}{6}$ and $|f(z_1)| \geq e|f(z_2)|$, we have (2.6).

Proof. We take the geodesic curve γ connecting z_1 and z_2 in U. We may assume that $|f(z_2)| \leq |f(z)| \leq |f(z_1)|, \ \forall\ z \in \gamma$. Set $\Omega = f(U)$. In view of Lemma 1, we have

$$\lambda_U(z) \geq \lambda_\Omega(f(z))|f'(z)| \geq \frac{1}{\beta_\Omega(f(z);0) + \kappa}\frac{|f'(z)|}{|f(z)|},$$

which reduces to

$$(\beta_\Omega(f(z);0) + \kappa)\lambda_U(z) \geq \frac{|f'(z)|}{|f(z)|}.$$

There exits a point $\hat{z} \in \gamma$ such that $\beta_\Omega(f(z);0) \leq \beta_\Omega(f(\hat{z});0), \ \forall\ z \in \gamma$. Considering the integration along γ yields

$$(\beta_\Omega(f(\hat{z});0) + \kappa)d_U(z_1, z_2) \geq \int_\gamma \frac{|f'(z)|}{|f(z)|}|dz| \geq \left|\int_\gamma \frac{f'(z)}{f(z)}dz\right| \geq \log\frac{|f(z_1)|}{|f(z_2)|}.$$

Therefore we have

$$\beta_\Omega(f(\hat{z});0) \geq \log e^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}.$$

In terms of the definition of β_Ω, we have

$$\Omega \supset \left\{ z:\ e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{1/\delta}|f(\hat{z})| < |z| < e^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}|f(\hat{z})| \right\}.$$

This is (2.5).

Suppose that $|f(z_1)| \geq \exp\left(\frac{\kappa\delta}{1-\delta}\right)|f(z_2)|$ and $0<\delta<1$. Then

$$e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{\frac{1-\delta}{\delta}} \leq 1,$$

and we have

$$e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{\frac{1}{\delta}}|f(\hat{z})| \leq e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{\frac{1}{\delta}}|f(z_1)|$$

$$= e^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{\frac{1-\delta}{\delta}}|f(z_2)| \leq |f(z_2)|,$$

and

$$e^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}|f(\hat{z})| \geq |f(z_1)|.$$

Thus (2.6) follows from (2.5).

If $\delta \leq \frac{1}{6}$ and $|f(z_1)| \geq e|f(z_2)|$, then $\delta < \frac{1}{\kappa+1}$ and $\frac{\kappa\delta}{1-\delta} < 1$ and $|f(z_1)| \geq e|f(z_2)| \geq \exp\left(\frac{\kappa\delta}{1-\delta}\right)|f(z_2)|$. This implies (2.6). □

Theorem C (Zheng). *Let f be an analytic covering mapping from a hyperbolic domain U onto an annulus $A := \{z \in \boldsymbol{C} : r < |z| < R\}$. Then for $z \in U$ and $a \notin U \cup \{\infty\}$, one of the following statements holds:*

(1) if a is in the unbounded component of $\mathbb{C} \setminus U$, then

$$|f'(z)| \geq \frac{\mathrm{mod}(A)}{2\pi}\frac{|f(z)|}{|z-a|}\sin\frac{\pi(\log R - \log|f(z)|)}{\mathrm{mod}(A)}$$

(2) if a is in the bounded component of $\mathbb{C} \setminus U$, then

$$|f'(z)| \geq \frac{2\min\{\mathrm{mod}(A), m\}}{(2\kappa+1)\pi}\frac{|f(z)|}{|z-a|}\sin\frac{\pi(\log R - \log|f(z)|)}{\mathrm{mod}(A)}, \tag{2.7}$$

where m is the covering number of $f(z)$ from U onto A.

3. Nevanlinna Theory

Set $\log^+ x = \log \max\{1, x\}$. Let $f(z)$ be a meromorphic function. Define

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, r) \log r,$$

$n(t, f)$ — the number of poles of f in $\{z : |z| < t\}$, counted with multiplicities, and

$$T(r, f) := m(r, f) + N(r, f),$$

as the Nevanlinna characteristic of f.

The Nevanlinna-Jesen formula:

$$T(r, f) = T\left(r, \frac{1}{f}\right) + \log |c(0)|,$$

that is,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = N\left(r, \frac{1}{f}\right) - N(r, f) + \log |c(0)|,$$

where $c(0)$ is the first coefficient of Laurent series of f at 0.

The Nevanlinna deficiency of f at ∞:

$$\delta(\infty, f) := \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)}.$$

for $a \neq \infty$, $\delta(a, f)$ is defined by the above equation with $m\left(r, \frac{1}{f-a}\right)$ in the place of $m(r, f)$.

If $\delta(\infty, f) > 0$, for all sufficiently large r, we always have

$$m(r, f) > \frac{\delta(\infty, f)}{2} T(r, f)$$

4. Application of hyperbolic metric

Theorem D (Zheng). *Let $f(z)$ be a transcendental meromorphic function with finitely many poles. If U is a Fatou component of f such that $f^n|_U \to \infty (n \to \infty)$ and for all sufficiently large n, U_n separates 0 and ∞, then for all $n > n_0$, U_n contains an annulus $\{z : r_n < |z| < R_n\}$ with $r_n \to \infty$ and $\frac{R_n}{r_n} \to \infty$.*

Theorem E (Bergweiler, Rippon and Stallard). *Let $f(z)$ and U are given as in Theorem D. Then for $z_0 \in U$ and an open set D in U containing z_0, there exists a $\alpha > 0$ such that for all sufficiently large n we have*

$$U_n \supset f^n(D) \supset \{z : |f^n(z_0)|^{1-\alpha} < |z| < |f^n(z_0)|^{1+\alpha}\}.$$

In fact, for all sufficiently large r one can take a $z_0 \in U$ such that for some $p > 0$, $|f^{n+p}(z_0)| \geq M_n(r, f),\ \forall\ n \in \mathbb{N}$, where $M_n(r, f)$ is the nth iterate of $M(r, f)$.

Bergweiler, Rippon and Stallard proved that if U is a multiply connected Fatou component of an entire function f, then for $z_0 \in U$ the limit

$$h_U(z) = \lim_{n\to\infty} \frac{\log |f^n(z)|}{\log |f^n(z_0)|}$$

exists on U and $h_U(z)$ is a positive non-constant harmonic function on U.

This implies that $h_U(z) \equiv 1$ on U if and only if U and U_n are simply connected.

By using the harmonic function $h_U(z)$ they characterized precisely the round annulus in U_n.

In [28], we proved that if $f^n|_U \to \infty$ for a Fatou component U, then for any compact subset W of U, there exists a $M(W) > 1$ such that

$$M(W)^{-1}|f^n(z)| \leq |f^n(w)| \leq M(W)|f^n(z)|, \ \forall \ z, w \in W \tag{4.1}$$

provided that $\cup_{n=1}^{\infty} f^n(U)$ does not contain any sequence of round annulus D_m centered at 0 such that $\mathrm{dist}(0, D_m) \to \infty$ and $\mathrm{mod}(D_m) \to \infty$.

Thus, (4.1) must hold if U is unbounded, and in particular, if U is a Baker domain, which is proved in [29] and Rippon [21].

In this case, $h_U(z) \equiv 1$ on U.

(4.1) does not hold for a multiply connected Fatou component U of an entire function.

Problem 2. *If (4.1) does not hold for a compact subset W of f, should $\cup_{n=1}^{\infty} f^n(U)$ contain a sequence of round annulus D_m centered at 0 such that* $\mathrm{dist}(0, D_m) \to \infty$ *and* $\mathrm{mod}(D_m) \to \infty$?

Theorem 2. (Zheng) *Let $f(z)$ be a transcendental meromorphic function. Assume that there exist two points a, b in a Fatou component U such that we have*

$$\frac{|f^{n_k}(a)|}{|f^{n_k}(b)|} \to \infty, \ |f^{n_k}(b)| \to \infty (k \to \infty). \tag{4.2}$$

(i) if for all sufficiently large r,

$$n(r, 0) \geq n(r, \infty) + 6\sqrt{2}\pi,$$

then for all sufficiently large n, U_n contains annulus $A_n = \{z : r_n < |z| < R_n\}$ and $A_{n+1} \subset f(A_n)$ and $r_n \to \infty$ and $\frac{R_n}{r_n} \to \infty$.

(ii) if for all sufficiently large r,

$$T(r, f) \geq N(r, f) \log r,$$

then for all sufficiently large n, U_n contains annulus $A_n = \{z : r_n < |z| < r_n^{\alpha}\}$ with $\alpha > 1$ and $A_{n+1} \subset f(A_n)$ and $r_n \to \infty$.

Theorem 3. (Zheng) *Let $f(z)$ be a transcendental meromorphic function such that*

$$T(2r,f) \geq dT(r,f),\ \ r \geq r_0,\ \ d > 1$$

and $\delta(\infty, f) > 0$. Then for any compact subset W in a Fatou component of f, we have (4.1) for a $M(W) > 1$.

Example 1. *There exists a meromorphic function which has a multiply connected wandering domain U such that $f^n|_U \to \infty(n \to \infty)$ and every U_n separates 0 and ∞ with $\sup_n \mathrm{Mod}_0(U_n) < \infty$.*

Example 2. *There exists a transcendental meromorphic function $f(z)$ which has a wandering domain U such that U_n contains a round annulus D_n centered at 0 and $\mathrm{mod}(D_n) \to \infty(n \to \infty)$ and there are two points $a, b \in U$ such that $\frac{|f^n(a)|}{|f^n(b)|} \to \infty(n \to \infty)$ but $h_U(z) \equiv 1$ on U.*

Example 3. *There exists a transcendental meromorphic function $f(z)$ which has a wandering domain U such that U_n contains a round annulus D_n centered at 0 and $\mathrm{mod}(D_n) \to \infty(n \to \infty)$ and there are two points $a, b \in U$ such that $\frac{|f^n(a)|}{|f^n(b)|} \to \infty(n \to \infty)$ but $h_U(z)$ does not exist on U.*

Problem 3. *Under Case (ii) of Theorem 1, does $h_U(z)$ exist?*

That is to say, if we confine suitably the number of poles, whether does $h_U(z)$ exist?

5. Proof of Theorem 2

Theorem 2 (Zheng) *Let $f(z)$ be a transcendental meromorphic function. Assume that there exist two points a, b in a Fatou component U such that we have*

$$\frac{|f^{n_k}(a)|}{|f^{n_k}(b)|} \to \infty,\ |f^{n_k}(b)| \to \infty (k \to \infty). \tag{5.1}$$

(i) if for all sufficiently large r,

$$n(r, 0) \geq n(r, \infty) + 6\sqrt{2}\pi,$$

then for all sufficiently large n, U_n contains annulus $A_n = \{z : r_n < |z| < R_n\}$ and $A_{n+1} \subset f(A_n)$ and $r_n \to \infty$ and $\frac{R_n}{r_n} \to \infty$.

(ii) if for all sufficiently large r,

$$T(r, f) \geq N(r, f) \log r,$$

then for all sufficiently large n, U_n contains annulus $A_n = \{z : r_n < |z| < r_n^\alpha\}$ with $\alpha > 1$ and $A_{n+1} \subset f(A_n)$ and $r_n \to \infty$.

Proof. Under the assumption of Theorem 2, in view of a result in [29], U is not a Baker domain and it is a wandering domain.

Given arbitrarily a $C \geq e^2$, under the assumption of Theorem 2 we can take a m such that $e^{-\kappa}\left(\frac{|f^m(a)|}{|f^m(b)|}\right)^{1/\delta} \geq C^d$, where $\delta = d_U(a, b)$ and $d = 6\sqrt{2}\pi$ and $|f^m(b)|$ is sufficiently large such that for $r \geq |f^m(b)|$ the inequalities in (i) and (ii) hold.

Applying Theorem 1 to f^m on U yields

$$f^m(U) \supset \{z :\ C^{-d} r_0 < |z| < C^d r_0\},$$

where $|f^m(b)| \leq r_0 \leq |f^m(a)|$.

Let us prove Part (i) of Theorem 2. Under the condition of Part (i), we have

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right)+\log|f(0)|\geq N\left(r,\frac{1}{f}\right)+\log|f(0)| \\ &\geq N(r,f)+d\log r+O(1). \end{aligned}$$

That is

$$m(r,f)\geq d\log r+O(1).$$

Therefore for sufficiently large r_0 we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(r_0e^{i\theta})|d\theta=m(r_0,f)\geq\log r_0.$$

There exists a point z_0 with $|z_0|=r_0$ such that $|f(z_0)|\geq r_0$. Since $f(f^m(U))\cap f^m(U)=\emptyset$, we have $|f(z)|>1$ on $f^m(U)$.

Set $A=\{z:\ C^{-d}r_0<|z|<C^dr_0\}$ and $B=\{z:C^{-1}r_0<|z|<Cr_0\}$.

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Cr_0e^{i\theta})|d\theta=T(Cr_0,f)-N(Cr_0,f)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(C^{-1}r_0e^{i\theta})|d\theta=T(C^{-1}r_0,f)-N(C^{-1}r_0,f),$$

there exist two points z_1 and z_2 with $|z_1|=Cr_0$ and $|z_2|=C^{-1}r_0$ such that

$$\log|f(z_1)|\geq T(Cr_0,f)-N(Cr_0,f)$$

and

$$\log|f(z_2)|\leq T(C^{-1}r_0,)-N(C^{-1}r_0,f).$$

Noting that

$$\begin{aligned} & T(Cr_0,f)-N(Cr_0,f)-T(C^{-1}r_0,f)+N(C^{-1}r_0,f) \\ = & N\left(Cr_0,\frac{1}{f}\right)-N(Cr_0,f)-N\left(C^{-1}r_0,\frac{1}{f}\right)+N(C^{-1}r_0,f) \\ \geq & \int_{C^{-1}r_0}^{Cr_0}\frac{n(t,0)-n(t,\infty)}{t}dt\geq\int_{C^{-1}r_0}^{Cr_0}\frac{d}{t}dt=2d\log C, \end{aligned}$$

we have

$$|f(z_1)|\geq C^{2d}|f(z_2)|.$$

Since $\lambda_A(z)\leq\frac{\pi\sqrt{2}}{|z|4d\log C}$, $\forall z\in B$ with $d=6\sqrt{2}\pi$ and $C\geq e^2$, we have

$$\rho_A(z_1,z_2)\leq\frac{\pi\sqrt{2}}{2d}+\frac{\sqrt{2}\pi^2}{4d\log C}<\frac{1}{6}.$$

In terms of Theorem 1 we have

$$f^{m+1}(A) \supset \{z;\ |f(z_2)| < |z| < |f(z_1)|\}. \tag{5.2}$$

There exists a $r_1 > 0$ such that

$$f^{m+1}(U) \supset f^{m+1}(A) \supset \{z;\ C^{-d}r_1 < |z| < C^d r_1\},$$

and $C^{-d}r_1 \geq C^d r_0,\ r_1 > C^{2d} r_0$.

Then we can continue the above step and inductively we have

$$f^{m+n}(U) \supset \{z;\ C^{-d}r_n < |z| < C^d r_n\}.$$

and $r_n \geq C^{2nd} r_0 \to \infty (n \to \infty)$.

Since C can be chosen be larger and larger as m increases, we have proved the existence of the annuli $A_n = \{z; r_n < |z| < R_n\}$ with $R_n/r_n \to \infty (n \to \infty)$ such that $A_n \subset U_n$ and $A_{n+1} \subset f(A_n)$.

Now we prove Part (ii) of Theorem 2. Set $\hat{T}(x) = e^{T(x,f)}$. Under the assumption of Part (ii) of Theorem 1, for all sufficiently large r we have

$$\begin{aligned} T(Cr,f) - N(Cr,f) &\geq \left(1 - \frac{1}{\log Cr}\right) T(Cr,f) \\ &\geq \left(1 - \frac{1}{\log Cr}\right)\left(1 + \frac{\log C/2}{\log 2r}\right) T(2r,f) \\ &\geq T(2r,f) + \frac{1}{2}\log\frac{C}{2}\frac{T(2r,f)}{\log 2r} \\ &\geq T(2r,f) + d\log C, \end{aligned} \tag{5.3}$$

where we have used the fact that $T(r,f)/\log r \to \infty (r \to \infty)$ for a transcendental meromorphic function f, and

$$T(C^{-1}r,f) - N(C^{-1}r,f) \leq T(C^{-1}r,f) \leq T(r,f) - d\log C. \tag{5.4}$$

It follows from (5.2) that

$$f^{m+1}(U) \supset \{z; e^{T(C^{-1}r_0,f) - N(C^{-1}r_0,f)} < |z| < e^{T(Cr_0,f) - N(Cr_0,f)}\}$$

$$\supset \{z : C^{-d}\hat{T}(r_0) < |z| < C^d \hat{T}(2r_0)\}.$$

For the next step, set $A_1 = \{z : C^{-d}r_1 < |z| < C^d R_1\}$ and $B_1 = \{z : C^{-1}r_1 < |z| < CR_1\}$ with $r_1 = \hat{T}(r_0)$ and $R_1 = \hat{T}(2r_0)$. Thus inductively we have

$$f^{m+n}(U) \supset A_n,\ A_n = \{z :\ C^{-d}\hat{T}_n(r_0) < |z| < C^d \hat{T}_n(2r_0)\}.$$

A similar calculation yields that $\hat{T}_n(2r_0) = \hat{T}_n(r_0)^{1+\log 2/\log r_0}$. Therefore, we have proved that for all sufficiently large n, $f^n(U)$ contains the annulus $\{z : T_n < |z| < T_n^c\}$ with $c > 1$ and $T_n \to \infty (n \to \infty)$. $\square$

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