What is the next number in the sequence?
(or, Sublattice counting and orbifolds)

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Gauge theories: quivers, tilings and Calabi–Yaus — 14 May 2014

Collaboration with: A. Hanany (Imperial College), S. Reffert (CERN).
Outline

1. Why are we here?
2. Brane tilings
3. Symmetries
   - Cycle index
   - Burnside’s lemma
   - Hermite normal form
4. Some number theory
5. Generating functions
   - Power series and Dirichlet series
   - Dirichlet convolution
   - Asymptotic behaviour
6. Conclusions
Motivation

- Brane tilings have met with a lot of interest in the past few years.
- A brane tiling is the dual description of a quiver gauge theory,
- D3 branes probing a toric Calabi–Yau-three singularity
- M2 branes probing a toric Calabi–Yau-four singularity
- We would like to have a classification of all possible tilings.
Today’s talk

- For the moment a complete classification seems too ambitious
- We consider a simpler problem: counting Abelian orbifolds of a given theory
- An orbifold is described by a repetition of the fundamental domain
- This is the definition of a sublattice of the initial lattice
- We map our orbifold counting problem to a sublattice enumeration. We can use methods from crystallography
- We find a number theoretical description for the generating functions. Asymptotically

\[
f(n) \sim \frac{\sigma(n)}{|G|}
\]
How do you do that?

- Find the **symmetries** of a given lattice (cycle index)
- **Decompose** the counting function on the symmetries (Burnside’s lemma)
- **Count** the sublattices for each symmetry (Hermite normal form)
- Study the **structure** (Dirichlet convolution)
- Write in a **compact form** (Dirichlet series)
- **Analysis** (Asymptotic behaviour)
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6 Conclusions
Consider a Calabi-Yau three-fold $X$ and let $S \subset X$ be a surface shrinking to a point. Placing $D$-branes on $S$ we expect:

- An enhanced gauge symmetry because some open strings will shrink to zero length.
- The branes are marginally stable against the decay into fractional branes.

These fractional branes are rigid branes generating the BPS states in the theory as bound states.

The description of the brane as a manifold breaks down.

Geometric description: language of categories.

For our purposes we can use quiver gauge theories.
**D-branes at singularities**

- **Quiver on the torus**
  - A → B → C → D → A

- **Graph dual**
  - D → C → B → A → D

- **Brane tiling**
  - D → C → B → A → D

- **Toric diagram**

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What is the next number in the sequence?
Let us consider $\mathbb{Z}_n$ orbifolds of $\mathbb{C}^3$.

Let us denote the coordinates of $\mathbb{C}^3$ by $\{z_1, z_2, z_3\}$, and the orbifold action by $(a_1, a_2, a_3)$:

$$\{ z_1, z_2, z_3 \} \sim \{ \omega^{a_1} z_1, \omega^{a_2} z_2, \omega^{a_3} z_3 \}$$

with $\omega^n = 1$ and $a_1 + a_2 + a_3 = 0 \mod n$.

In this notation, the problem is to find all triples $(a_1, a_2, a_3)$ that give inequivalent orbifolds of $\mathbb{C}^3$. 

What is the next number in the sequence?
An alternative way of formulating the problem is by looking at the toric diagrams of these orbifolds.

The toric diagram of $\mathbb{C}^3$ is a triangle of unit area.

The toric diagram of an orbifold of $\mathbb{C}^3$ by an Abelian group of order $n$ is again a triangle but with an area which is $n$ times larger.

The problem of counting all inequivalent orbifolds of $\mathbb{C}^3$ is therefore equivalent to the problem of finding all triangles with vertices on integral points and area $n$.

Since these are toric diagrams, two triangles which are related by a $GL(2, \mathbb{Z})$ are equivalent.
Orbifolds: brane tilings

- Think of the brane tiling as forming a **bipartite hexagonal lattice**
- the problem of finding inequivalent toric diagrams is mapped to the problem of finding its **sublattices**

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>brane tiling</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
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<td>geometry</td>
<td>$\mathbb{C}^3$</td>
<td>$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$</td>
<td>$\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$</td>
<td>$\mathbb{C}^3/\mathbb{Z}_3$</td>
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</tbody>
</table>

What is the next number in the sequence?
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What is the next number in the sequence?
We need a way to capture the symmetries of a given lattice. 

- Label the vertices of the fundamental cell by the numbers \( \{ 1, \ldots, m \} \).
- We want to describe the group of permutations \( G \) of the set \( X = \{ 1, \ldots, m \} \) which result in the same fundamental cell.

![Diagram of a triangle with vertices labeled 1, 2, and 3]

- identity
- 3 reflections
- 2 rotations by \( 2\pi /3 \)

**Cycle notation**

What is the next number in the sequence?
Cycle notation

- Cycles of $g \in G$ are the orbits of the elements $\varepsilon \in X$ under $g$.
- For each group element $g$ we start with $\varepsilon_1 \in X$ and write down its orbit in parentheses, $(\varepsilon_1 \ g(\varepsilon_1) \ g^2(\varepsilon_1) \ldots \ g^{k-1}(\varepsilon_1))$, where $g^k(\varepsilon_1) = \varepsilon_1$.
- We continue with the next element that has not yet appeared in an orbit until we have exhausted all the elements of $X$.
- Each $g \in G$ can be expressed in terms of $\alpha_k$ disjoint cycles of length $k$.
- The type of $g$ is given by the partition of $m$ [1 $\alpha_1$ 2 $\alpha_2$ $\ldots$ $l \alpha_l$], where $m = \alpha_1 + 2 \alpha_2 + \ldots + l \alpha_l$.
- The partition is represented by the expression

$$\zeta_g(x_1, \ldots, x_l) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_l^{\alpha_l}.$$
## Cycle index

<table>
<thead>
<tr>
<th>$g$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$c(\alpha_i)$</th>
<th>$\zeta$</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>$x_1^3$</td>
</tr>
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<td>(12)</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>$x_1x_2$</td>
</tr>
<tr>
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<td>(123)</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>2</td>
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</tbody>
</table>

### The cycle index of $G$

$$Z_G(x_1, \ldots, x_l) = \frac{1}{|G|} \sum_{g \in G} \zeta_g(x_1, \ldots, x_l) = \frac{1}{|G|} \sum_{\alpha} c(\alpha_1, \ldots, \alpha_l) x_1^{\alpha_1} \cdots x_l^{\alpha_l}$$
Cycle index, examples

- **Cyclic group** $C_m$: symmetries of a circular object without reflections:

  \[ Z(C_m) = \frac{1}{m} \sum_{d|m} \varphi(d)x^{m/d}_d, \]

  where $\varphi(d)$ is the totient function.

- **Dihedral group** $D_m$: symmetries of a circular object:

  \[ Z(D_m) = \frac{1}{2}Z(C_m) + \begin{cases} \frac{1}{2}x_1x^{(m-1)/2}_2, & \text{if } m \text{ is odd}, \\ \frac{1}{4}\left(x_1^2x^{(m-2)/2}_2 + x^{m/2}_2\right), & \text{if } m \text{ is even}. \end{cases} \]

- The **symmetric group** $S_m$ is the group of all permutations:

  \[ Z(S_m) = \sum_{\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = m} \frac{1}{k!} \prod_{k=1}^{m} x^{\alpha_k}_k. \]
Burnside’s lemma

Let $G$ be a group of permutations of the set $X$. The number $N(G)$ of orbits of $G$ is given by the average over $G$ of the sizes of the fixed sets:

$$N(G) = \frac{1}{|G|} \sum_{g \in G} |F_g| ; \quad F_g = \{ x \in X \mid g(x) = x \}.$$ 

- The number $f(n)$ of sublattices of index $n$ is the number of orbits of the symmetry group $G$ when acting on the set $X_n$ of sublattices of index $n$.
- This can be written as the average of the number of elements in $X_n$ that are left invariant by the action of $g \in G$:

$$f(G) = \frac{1}{|G|} \sum_{g \in G} f_g(n) ; \quad f_g(n) = |\{ x \in X_n \mid g(x) = x \}|$$
Decomposition of the counting function

- Using the cycle decomposition we can rewrite this expression as a sum over the types of the elements \( g \), indexed by partitions \( \alpha \):

\[
 f(n) = \frac{1}{|G|} \sum_{\alpha} c(\alpha) f_{x^\alpha}(n) .
\]

- there is a subsequence for each monomial in the cycle index \( Z_G \).

\[
 Z_G(x_1, \ldots, x_l) = \frac{1}{|G|} \sum_{\alpha} c(\alpha) x_1^{\alpha_1} \cdots x_l^{\alpha_l}
\]

- we have decomposed the counting problem into subproblems fixed by the symmetry groups.

**Recipe**

For a given lattice, find the symmetry group \( G \), write the cycle index \( Z_G \) and identify the counting functions for each of the terms.
The hexagonal lattice

- Consider the bipartite hexagonal lattice corresponding to the geometry of $C^3$.
- Because of the bipartiteness, the symmetry group is $S_3$ (equilateral triangle).
- From the cycle decomposition above:

$$Z_{S_3} = \frac{1}{6} \left( x_1^3 + 3x_1x_2 + 2x_3 \right).$$

- Using Burnside’s lemma, the number of sublattices of index $n$ can be decomposed as:

$$f^\triangle(n) = \frac{1}{6} \left( f^\triangle_{x_1^3}(n) + 3f^\triangle_{x_1x_2}(n) + 2f^\triangle_{x_3}(n) \right)$$

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The hexagonal lattice

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
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<tr>
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<td>(f_{x_3}^\triangle)</td>
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</tr>
</tbody>
</table>

\(f^\triangle\) Scatter plot of the sequence \(f^\triangle\) for a hexagonal lattice. Prime numbers are emphasized in red. The two lines correspond to \(n/6\) and \(e^{r_n \log \log n/6}\).
Hermite normal form

- The problem of counting sublattices with a given symmetry can be mapped to the problem of counting a set of matrices.
- We learn what is the general structure.
- We obtain an algorithm to count the sublattices.
- Consider a lattice $L_d$ generated by the $d$ vectors $\langle y_1, \ldots, y_d \rangle$. Any sublattice $L'$ of $L_d$ is generated by $d$ vectors $\langle x_1, \ldots, x_d \rangle$

\[
\begin{align*}
x_1 &= a_{11}y_1 \\
x_2 &= a_{21}y_1 + a_{22}y_2 \\
&\vdots \\
x_d &= a_{d1}y_1 + a_{d2}y_2 + \cdots + a_{dd}y_d,
\end{align*}
\]

- The integer coefficients $a_{ij}$ satisfy the conditions

\[0 \leq a_{ij} < a_{ii} \quad \forall j < i \quad n = \prod_{i=1}^{d} a_{ii}.
\]
**Hermite normal form**

- For a two dimensional lattice the coefficients are \( \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \).
- From the condition \( a_{11}a_{22} = n \) we choose \( a_{22} = m \) and \( a_{11} = n/m \), where \( m \) is a divisor of \( n \).
- To count the number of sublattices invariant under the symmetry \( \mathbf{x}^{\alpha} \) we enumerate the possible values of \( a_{21} \).
- The constraint \( a_{21} < a_{22} \) introduces a dependence of the number of possible values of \( a_{21} \), \( \# \{ a_{21} \} = g_{\mathbf{x}^{\alpha}}(a_{22}) \), on \( a_{22} \).
- The total number of sublattices \( f_{\mathbf{x}^{\alpha}}(n) \) is given by summing \( g_{\mathbf{x}^{\alpha}}(m) \) over all the divisors of \( n \):
  \[ f_{\mathbf{x}^{\alpha}}(n) = \sum_{m|n} g_{\mathbf{x}^{\alpha}}(m). \]
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   - Conclusions

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6. **Conclusions**

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**What is the next number in the sequence?**
Multiplicative functions

- Prime numbers play a special role.
- This is one of the clues that point to the sequences being multiplicative.

Multiplicative sequence

A sequence $f$ is multiplicative if

$$f(nm) = f(n)f(m), \quad \text{when } (n, m) = 1,$$

where $(n, m)$ is the greatest common divisor between $n$ and $m$.

- $f$ is completely determined by its values for primes and their powers
- for any $n = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$, the sequence decomposes as

$$f(n) = f(p_1^{a_1})f(p_2^{a_2}) \ldots f(p_r^{a_r})$$

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What is the next number in the sequence?
Dirichlet convolution

The Dirichlet convolution of two sequences \( g \) and \( h \) is

\[
f(n) = (g * h)(n) = \sum_{m|n} g(m) h\left(\frac{n}{m}\right),
\]

where the sum runs over all the divisors \( m \) of \( n \).

- **Commutative**, \( f * g = g * f \),
- **Associative**, \( f * (g * h) = (f * g) * h \)
- Has an **identity** \( f * \text{Id} = f \) defined by
  \[
  \text{Id}(n) = \{1, 0, 0, \ldots\}
  \]
- to each sequence \( f \) one can associate its **inverse** \( f^{-1} \) satisfying
  \[
  f * f^{-1} = f^{-1} * f = \text{Id}.
  \]
We have seen that the counting functions have the structure

$$f_{x^\alpha}(n) = \sum_{m|n} g_{x^\alpha}(m).$$

In terms of Dirichlet convolution: $f_{x^\alpha} = g_{x^\alpha} * u$ where:

$$u(n) = \{1, 1, 1, \ldots\}.$$ 

its inverse is the Möbius function defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is square-free}, \\ 0 & \text{otherwise.} \end{cases}$$

where $k$ is the number of distinct prime factors of $n$.

It follows that if $f = g * u$, then $g = \mu * f$. 

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What is the next number in the sequence?
The hexagonal lattice

- The sequence $f^\triangle_{x^3_1} = \{ 1, 3, 4, 7, 6, 12, 8, 15, \ldots \}$ corresponds to the identity permutation $x^3_1$

  $$f^\triangle_{x^3_1} = u \ast N,$$

  where

  $$N(n) = \{ 1, 2, 3, \ldots \}.$$

- The sequence $f^\triangle_{x^1x^1_2} = \{ 1, 1, 2, 3, 2, 2, 2, 5, \ldots \}$ can be written as the convolution of a periodic sequence of period 4 and the unit:

  $$f^\triangle_{x^1x^1_2} = \{ 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots \} \ast u.$$

$g^\triangle_{x^1x^1_2}$ is in turn the convolution of a finite sequence and $u$:

$$f^\triangle_{x^1x^1_2} = \{ 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots \} \ast u = \{ 1, -1, 0, 2 \} \ast u \ast u.$$
The hexagonal lattice

The last sequence \( f_{x_3}^\Delta = \{ 1, 0, 1, 1, 0, 0, 2, 0, \ldots \} \) also has the form of the convolution of the unity with a periodic sequence of period 3:

\[
f_{x_3}^\Delta = \{ 1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots \} \ast u.
\]

The periodic sequence is the (non-principal) \textit{Dirichlet character} of modulus three:

\[
g_{x_3}^\Delta = \chi_{3,2}(n) = \{ 1, -1, 0, 1, -1, 0, \ldots \}.
\]

Putting all together we find that the sequence \( f^\Delta \) can be written as

\[
f^\Delta = \frac{1}{6} \left( f_{x_3}^\Delta + 3 f_{x_1x_2}^\Delta + 2 f_{x_3}^\Delta \right) = \frac{1}{6} \left( N + 3 \{ 1, 0, -1, 2 \} \ast u + 2 \chi_{3,2} \right) \ast u.
\]
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Generating functions

Now that we have the numbers we need a good way to encode them in a compact form.

- the **formal power series** (partition function)

\[
F(t) = \sum_{n=1}^{\infty} f(n) t^n ;
\]

- the **Dirichlet series**

\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} .
\]

- The corresponding **inverse transformations** are given by

\[
f(n) = \frac{1}{2\pi i} \oint \frac{F(t)}{t^{n+1}} \, dt , \quad f(n) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(s)n^s |_{s = \sigma + i \tau} \, d\tau .
\]
Generating functions and Dirichlet convolution

- Both types of generating functions have a **simple behavior under Dirichlet convolution**.
- Let $f, g$ and $h$ be such that
  
  $$f = g \ast h.$$  

- The power series for $h$ reads:

  $$F(t) = \sum_{n=1}^{\infty} f(n) t^n = \sum_{n=1}^{\infty} \sum_{m|n} g(m) h\left(\frac{n}{m}\right) t^n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} g(m) h(k) t^{mk}.$$  

  This can be expressed in two ways, using the generating function for $g$ or for $h$:

  $$F(t) = \sum_{m=1}^{\infty} g(m) H(t^m) = \sum_{k=1}^{\infty} h(k) G(t^k).$$
Generating functions and Dirichlet convolution

- All our sequences are sums over divisors (or equivalently as Dirichlet convolutions with the unit), we will always write

\[ F(t) = \sum_{k=1}^{\infty} G(t^k). \]

- It is also possible to write the power series for the inverse of the Dirichlet convolution as follows. Let

\[ f(t) = \sum_{k,m=1}^{\infty} g(m) h(k) t^{mk}, \]

then

\[ H(t) = \sum_{k=1}^{\infty} h(k) t^k = \sum_{m=1}^{\infty} \mu(k) g(k) F(t^k), \]

where \( \mu \) is the Möbius function.
The Dirichlet series is even more adapted to these structures.

- if \( f \) is multiplicative, the series can be expanded in terms of an infinite product over the primes, the Euler product:

\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots \right).
\]

remember that a multiplicative sequence is determined by the values taken for powers of prime numbers.

- the Dirichlet series of a convolution is decomposed as

\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \sum_{m|n} \frac{g(m) h\left(\frac{n}{m}\right)}{n^s} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{g(m) h(k)}{m^s k^s} = G(s)H(s).
\]

The Dirichlet series is the Laplace transform of a discrete measure. It exchanges convolution and pointwise products.
The hexagonal lattice

- The generating series are **linear transformations**. The decomposition in terms of symmetries remains the same.
- We can calculate explicitly the generating series for each term

<table>
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<tr>
<th>symmetry</th>
<th>Dirichlet series $G(s)$</th>
<th>power series $G(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^3$</td>
<td>$\zeta(s - 1)$</td>
<td>$\frac{1 + t^3}{(1 - t)(1 - t^2)} - 1$</td>
</tr>
<tr>
<td>$x_1x_2$</td>
<td>$(1 - 2^{-s} + 2^{1-2s})\zeta(s)$</td>
<td>$\frac{1 + t^3}{(1 - t)(1 + t^2)} - 1$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$L(s, \chi_{3,2})$</td>
<td>$\frac{1 + t^3}{1 - t^3} - 1$</td>
</tr>
</tbody>
</table>

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The hexagonal lattice

Collecting all the terms we find:

- For the power series:

\[
F^\Delta(t) = \sum_{m=1}^{\infty} \sum_{n_1,n_2,n_3=0}^{\infty} \sum_{n \neq (0,0,0)} (-1)^{n_2} t^{m(n_1+2n_2+3n_3)}.
\]

- For the Dirichlet series:

\[
F^\Delta(s) = \frac{\zeta(s)}{6} \left( \zeta(s-1) + 3 \left( 1 - \frac{1}{2^s} + \frac{2}{2^{2s}} \right) \zeta(s) + 2L(s, \chi, 3, 2) \right).
\]
Asymptotic behaviour

The asymptotic behavior of a sequence can be derived by looking at the corresponding Dirichlet series.

Asymptotic behaviour

Let $F(s)$ be a Dirichlet series with non-negative coefficients that converges for $\Re(s) > \alpha > 0$. If $F(s)$ is holomorphic in all points of the line $\Re(s) = \alpha$, except for $s = \alpha$ and

$$F(s) \sim A(s) + \frac{B(s)}{(s - \alpha)^{m+1}},$$

where $m \in \mathbb{N}$, then the partial sum of the coefficients is asymptotic to:

$$\sum_{n=1}^{N} f(n) \sim \frac{B(\alpha)}{\alpha \, m!} N^{\alpha} \log^{m}(N).$$
Asymptotic behaviour

We just need to know the analytic properties of some functions:

- The Riemann zeta function $\zeta(s)$ is analytic everywhere, except for a simple pole at $s = 1$ with residue 1;
- The $L$-function $L(s, \chi)$ is analytic everywhere, except for a simple pole at $s = 1$ if $\chi$ is a principal character.

Also useful is:

**Robin’s inequality**

$$\sigma(n) < e^{\gamma} n \log \log n, \quad n \text{ large},$$

where $\gamma$ is Euler’s constant. This is true for large $n$, where large means $n \geq 5041$, and if and only if Riemann’s hypothesis is true.
**Hexagonal lattice**

- The rightmost pole of the Dirichlet series $F^\Delta(s)$ is found for $s = 2$.
- The pole has order 1 and its residue is $\zeta(2)/6$.
- The partial sum of the terms in the sequence $f^\Delta$ behaves asymptotically as

\[
\sum_{n=1}^{N} f^\Delta(n) \sim \frac{\zeta(2)}{12} N^2 = \frac{\pi^2}{72} N^2,
\]

- for large $n$, the leading term is $\zeta(s)\zeta(s-1)/6$, hence

\[
f^\Delta(n) < \frac{e^\gamma n \log \log n}{6}, \quad n \text{ large.}
\]
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How do you do that?

- Find the **symmetries** of a given lattice (cycle index)
- **Decompose** the counting function on the symmetries (Burnside’s lemma)
- **Count** the sublattices for each symmetry (Hermite normal form)
- Study the **structure** (Dirichlet convolution)
- Write in a **compact form** (Dirichlet series)
- **Analysis** (Asymptotic behaviour)
We have seen the construction for the orbifolds of $\mathbb{C}^3$.

- the conifold

In the paper:

- $L^{aba}$
- $\mathbb{C}^4$

These exhausts all the symmetries in the plane

Classification of string vacua
Brane tilings describe quiver gauge theories obtained by placing branes at CY singularities.

We would like to obtain a complete classification.

We started by studying a subclass: Abelian orbifolds of a given geometry.

The problem is the same as counting sublattices of a given lattice.

Techniques from number theory.

Generating functions for any given geometry.
The end

Thank you for your attention