Cohen–Macaulay modules and Calogero–Moser systems

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Matrix factorizations and related topics
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- $\Phi \subset \mathbb{C}$ is a finite subset such that $\alpha - \beta \notin \pi\mathbb{Z}$ for any $\alpha \neq \beta \in \Phi$.
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Definition (Planar quasi–invariants)

The algebra of (Φ, μ)–quasi–invariant polynomials:

$$A = A\big(\Phi, \mu\big) := \Big\{ f \in R \;\big|\; l_\alpha^{2\mu_\alpha+1} \text{ divides } \big(f - s_\alpha(f)\big) \text{ for all } \alpha \in \Phi \Big\},$$

where $l_\alpha(z_1, z_2) := -\sin(\alpha) z_1 + \cos(\alpha) z_2 \in R := \mathbb{C}[z_1, z_2]$.

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Definition (Generalized divisor class group)

$$\mathsf{CM}_1^{\mathrm{lf}}(A) = \left\{ M \in A - \mathsf{mod} \;\middle|\; \begin{array}{l} M \cong M^{\vee\vee} \quad \mathsf{rk}_A(M) = 1 \\ M_{\mathfrak{m}} \cong A_{\mathfrak{m}} \text{ for all } \mathfrak{m} \in \mathsf{Max}(A) \setminus \Sigma_M \end{array} \right\}$$

$$M_1 \boxtimes_A M_2 := \bigl(M_1 \otimes M_2\bigr)^{\vee\vee}.$$

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where the group law $\circ$ on $K_\alpha = \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ is given by the rule

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Idea of the proof: for $\vec{\gamma} = \big(\gamma_\alpha\big)_{\alpha \in \Phi} \in K\big(\Phi, \mu\big)$, the corresponding module $M(\vec{\gamma}) \in \mathsf{CM}_1^{\mathrm{lf}}(A)$ is defined by the triple $\big(R, \oplus_{\alpha \in \Phi} K_\alpha, (1 + \varepsilon\gamma_\alpha)_{\alpha \in \Phi}\big)$.

$$M(\vec{\gamma}') \boxtimes_A M(\vec{\gamma}'') \cong M(\vec{\gamma}' \circ \vec{\gamma}'').$$

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where

$$\begin{cases} T^{+}_{(\alpha, 2\mu_\alpha)}(f) & = & \displaystyle\sum_{j=0}^{\mu_\alpha - 1} f_\alpha^{(2j)} \frac{\sigma^j}{(2j)!} \\ T^{-}_{(\alpha, 2\mu_\alpha)}(f) & = & \displaystyle\sum_{j=0}^{\mu_\alpha - 1} f_\alpha^{(2j+1)} \frac{\sigma^j}{(2j+1)!}. \end{cases}$$

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Moreover, for any $M \in \mathsf{CM}_1(A)$ *there exists* $\Phi \xrightarrow{\nu} \mathbb{N}_0$ *satisfying* $\nu_\alpha \le \mu_\alpha$ *for all* $\alpha \in \Phi$ *and* $N \in \mathsf{CM}_1^{\mathrm{lf}}\big(A(\Phi, \nu)\big)$ *such that* $M \cong N$.

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We first construct a certain (natural) group homomorphism

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We have: $\mathsf{Pic}(A) \cong \mathrm{Im}(\Upsilon) \cap \big(\prod_{\alpha \in \Phi} \mathbb{C}[\rho][\sigma]/(\sigma^{\mu_\alpha})\big) \subset \mathsf{CM}_1^{\mathrm{lf}}(A)$.

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Example

Let $A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3] \cong \mathbb{C}[z_1^2, z_1^3] \otimes_{\mathbb{C}} \mathbb{C}[z_2^2, z_2^3]$. Then we have:

$$\begin{array}{ccc}
\mathsf{CM}_1^{\mathsf{lf}}(A) & \xrightarrow{\quad\cong\quad} & \big(\mathbb{C}(\rho), +\big) \oplus \big(\mathbb{C}(\rho), +\big) \\
\uparrow & & \uparrow \\
\mathsf{Pic}(A) & \xrightarrow{\quad\cong\quad} & \Big\{(\rho f, \rho g) \in \rho\mathbb{C}[\rho] \oplus \rho\mathbb{C}[\rho] \;\Big|\; f'(0) + g'(0) = 0\Big\}.
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Let $\Phi = \left\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\right\}$, $\mu_\alpha = m$ for all $\alpha \in \Phi$ and $\vec{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$.

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \sum_{\alpha \in \Phi} \frac{m(m+1)}{l_\alpha^2(\vec{x} - \vec{\xi})}$$

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Recall: there exists an injective algebra homomorphism

$$\mathbb{C}[z_1, z_2] \supset A(\Phi, m) \xrightarrow{L} \mathfrak{D} := \mathbb{C}[\![x_1, x_2]\!][\partial_1, \partial_2], \quad z_1^2 + z_2^2 \mapsto H.$$

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$$F \cong \left\{ f \in \mathbb{C}[z_1, z_2] \,\middle|\, \exp(-\xi_1 z_1 - \xi_2 z_2) f \textit{ is } (\Phi, \mu) - \textit{quasi–invariant} \right\}.$$

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$$\Psi(\vec{x}, \vec{z}, \vec{\xi}) := \Big(H_{(x_1,x_2)} - z_1^2 - z_2^2\Big)^\kappa \circ \big(\delta(x_1 - \xi_1, x_2 - \xi_2) \cdot \exp(x_1 z_1 + x_2 z_2)\big),$$

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We have: $H_{(x_1,x_2)} \circ \Psi(x_1, x_2; z_1, z_2; \vec{\xi}) = (z_1^2 + z_2^2) \cdot \Psi(x_1, x_2; z_1, z_2; \vec{\xi})$.

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Moreover, there exists $A\big(\Phi, m\big) \xrightarrow{\Xi} \mathbb{C}(x_1, x_2)[\partial_1, \partial_2]$ *such that*

$$\big(\Xi(f)\big)_{(x_1,x_2)} \circ \Psi(x_1, x_2; z_1, z_2; \vec{\xi}) = f(z_1, z_2) \cdot \Psi(x_1, x_2; z_1, z_2; \vec{\xi}).$$

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for any $f \in A\big(\Phi, \mu\big) \subset \mathbb{C}[z_1, z_2]$. *In particular,* $H_{(x_1,x_2)} = \Xi(z_1^2 + z_2^2)$.

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For any $p_1, p_2 \in \mathbb{N}_0$ put

$$w_{(p_1,p_2)}(z_1,z_2) := \left.\frac{\partial^{p_1+p_2}\Psi(x_1,x_2,z_1,z_2;\vec{\xi})}{\partial x_1^{p_1}\partial x_2^{p_2}}\right|_{(x_1,x_2)=(0,0)} \in \mathbb{C}[z_1,z_2]$$

and $W := \left\langle w_{p_1,p_2} \middle| (p_1,p_2) \in \mathbb{N}_0 \times \mathbb{N}_0 \right\rangle \subset \mathbb{C}[z_1,z_2]$.

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The vector space W is an A–module, isomorphic to the spectral module $F := \mathfrak{D}/(x_1, x_2)\mathfrak{D} \cong \mathbb{C}[\partial_1, \partial_2] = \mathbb{C}[z_1, z_2]$ of the Calogero–Moser system.

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Finally,

$$\dim_{\mathbb{C}}\left\{ w \in W \;\middle|\; \deg(w) \le n + \kappa \right\} = \frac{(n+1)(n+2)}{2} \quad \text{for } n \in \mathbb{N}_0.$$

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where $(\xi_1, \xi_2) \in \mathbb{R}^2$ is such that $\xi_1 \xi_2 \neq 0$. In this case we have:

- $A = A(\Phi, m) = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, where $\Phi = \left\{0, \frac{\pi}{2}\right\}$ and $m = 1$.

Example: Calogero–Moser operator with a split potential

Consider the Calogero–Moser operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2\left(\frac{1}{(x_1 - \xi_1)^2} + \frac{1}{(x_2 - \xi_2)^2}\right),$$

where $(\xi_1, \xi_2) \in \mathbb{R}^2$ is such that $\xi_1 \xi_2 \neq 0$. In this case we have:

- $A = A(\Phi, m) = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, where $\Phi = \left\{0, \frac{\pi}{2}\right\}$ and $m = 1$.
- The spectral module F of the corresponding Calogero–Moser system has the following description:

$$F = \left\{ f \in \mathbb{C}[z_1, z_2] \;\middle|\; \begin{array}{l} \frac{\partial f}{\partial z_1}(0, \rho) = \xi_1 \rho f(0, \rho) \\ \frac{\partial f}{\partial z_2}(\rho, 0) = \xi_2 \rho f(\rho, 0) \end{array} \right\}.$$

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For any $\beta \in \mathbb{C}$, consider the vector space

$$W_\beta = \left\{ f \in R \;\middle|\; \begin{array}{l} \dfrac{\partial f}{\partial z_1}(0,\rho) = \dfrac{\xi_1^2 \rho}{\xi_1 + \beta\rho} f(0,\rho) \\ \dfrac{\partial f}{\partial z_2}(\rho,0) = \dfrac{\xi_2^2 \rho}{\xi_2 + \beta\rho} f(\rho,0) \end{array} \right\}.$$

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where $w = 1 + \xi_1 z_1 + \xi_2 z_2 + (\xi_1\xi_2 - \beta)\cdot\left(z_1 z_2 + \left(\dfrac{z_1^2}{\xi_2^2} + \dfrac{z_2^2}{\xi_1^2}\right)\right).$

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Theorem (Algebraic inverse scattering, Burban–Zheglov)

For any Schur pair (W, A), there exists an injective algebra homomorphism

$$A \xrightarrow{L_W} \mathfrak{S} := \left\{ \sum_{i_1, i_2 = 0}^{\infty} a_{i_1 i_2}(x_1, x_2)\partial_1^{i_1}\partial_2^{i_2} \;\middle|\; \inf_{(i_1, i_2)}\big(\upsilon(a_{i_1, i_2}) - (i_1 + i_2)\big) \in \mathbb{Z} \right\}.$$

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Summary

The study of Cohen–Macaulay modules with an appropriate Hilbert polynomial leads to new deformations of Calogero–Moser systems.

Thank you for your attention!