Matrix factorisations and orbifold Jacobian algebras

joint work with Alexey Basalaev (Heidelberg)
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Motivation

To a pair (f, G) of a polynomial f and a certain finite group G of symmetries of f, we want to associate an "orbifold version" of a Jacobian algebra in order to study an algebraic structure on the "first order deformation space of the pair (f, G)", the Hochschild cohomology group of the category of G-equivariant matrix factorizations of f.

Preliminaries

Definition 1

The *Jacobian algebra* $\mathrm{Jac}(f)$ of f is a $\mathbb{C}$-algebra defined as

$$\mathrm{Jac}(f) = \mathbb{C}[x_1, \ldots, x_N] \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right) .$$

Set

$$\Omega_f := \Omega^N(\mathbb{C}^N)/(df \wedge \Omega^{N-1}(\mathbb{C}^N)).$$

By choosing a nowhere vanishing N-form $\widetilde{\zeta} \in \Omega^N(\mathbb{C}^N)$ we have the following isomorphism

$$\vdash \zeta : \mathrm{Jac}(f) \xrightarrow{\cong} \Omega_f, \quad [\phi] \mapsto [\phi] \vdash \zeta := [\phi\widetilde{\zeta}],$$

where $\zeta := [\widetilde{\zeta}] \in \Omega_f$. Namely, Ω_f is a $\mathrm{Jac}(f)$-module of rank one.

Assume that $\dim_{\mathbb{C}} \mathrm{Jac}(f) < \infty$.

Proposition 2 (The residue pairing)

Define a $\mathbb{C}$-bilinear form $J_f : \Omega_f \otimes_{\mathbb{C}} \Omega_f \longrightarrow \mathbb{C}$ by

$$J_f(\omega_1, \omega_2) := \mathrm{Res}_{\mathbb{C}^N} \begin{bmatrix} \phi\psi dx_1 \wedge \cdots \wedge dx_N \\ \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_N} \end{bmatrix},$$

where $\omega_1 = [\phi dx_1 \wedge \cdots \wedge dx_N]$ and $\omega_2 = [\psi dx_1 \wedge \cdots \wedge dx_N]$. Then, the bilinear form J_f on Ω_f is non-degenerate.

In particular, J_f endows $\mathrm{Jac}(f)$ with a structure of a Frobenius algebra under $\mathrm{Jac}(f) \cong \Omega_f$ once we fix $\widetilde{\zeta}$.

Remark 3

An associative $\mathbb{C}$-algebra $(A, \circ)$ is called *Frobenius* if there is a non-degenerate $\mathbb{C}$-bilinear form $\eta : A \otimes A \to \mathbb{C}$ such that $\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$ for $X, Y, Z \in A$.

Aim

To a pair (f, G) of a polynomial f and a certain finite abelian group G of symmetries of f, associate a $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra $\mathrm{Jac}(f, G)$ which naturally generalize $\mathrm{Jac}(f)$.

Show the existence and uniqueness of the pair $(\mathrm{Jac}(f, G), \Omega_{f,G})$ which generalize $(\mathrm{Jac}(f), \Omega_f)$ with some expected properties.

Remark 4

Let $MF(f)$ be the dg category of matrix factorizations of f. Dyckerhoff shows

$$HH^\bullet(MF(f)) \cong \mathrm{Jac}(f), \quad HH_{\bullet+N}(MF(f)) \cong \Omega_f.$$

We are interested in the "calculus algebra" $(HH^\bullet, HH_\bullet)$ for G-equivariant (dg-)categories of matrix factorizations.

Invertible polynomials

Definition 5

A weighted homogeneous poly. $f \in \mathbb{C}[x_1, \ldots, x_N]$ is *invertible* if

1. f is non-degenerate and the number of variables coincides with the number of monomials in the polynomial f, namely,

$$f(x_1, \ldots, x_N) = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_j^{E_{ij}}, \quad c_i \in \mathbb{C}^*, \; E_{ij} \in \mathbb{Z}_{\geq 0}.$$

2. The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.

Remark 6

A weighted homogeneous polynomial f is called *non-degenerate* if it has at most an isolated critical point at the origin in $\mathbb{C}^N$.
One may assume that $c_i = 1$ for all i by rescaling the variables.

Definition 7 (Berglund–Hübsch transposition)

For an invertible polynomial

$$f(x_1, \dots, x_N) = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_j^{E_{ij}}, \quad c_i \in \mathbb{C}^*, \ E_{ij} \in \mathbb{Z}_{\geq 0},$$

the polynomial

$$\widetilde{f} := \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_j^{E_{ji}},$$

is called the *Berglund–Hübsch transpose* of f.

Proposition 8 (Kreuzer–Skarke)

An invertible polynomial f can be written as a Thom–Sebastiani sum $f = f_1 \oplus \cdots \oplus f_p$ of invertible polynomials f_ν, $\nu = 1, \ldots, p$ of the following types:

$$x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}, \quad \textit{chain type } (m \geq 1),$$
$$x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1, \quad \textit{loop type } (m \geq 2).$$

A chain type with $m = 1$, $x_1^{a_1}$, is also called the Fermat type.

From now on, $f = f(x_1, \ldots, x_N)$ denotes an invertible polynomial.

Group of symmetries of f

Definition 9

The *group of maximal diagonal symmetries* of f is defined as

$$
\begin{aligned}
G_f &:= \left\{ (\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1 x_1, \dots, \lambda_N x_N) = f(x_1, \dots, x_N) \right\} \\
&= \left\{ (\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \;\middle|\; \prod_{j=1}^{N} \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^{N} \lambda_j^{E_{Nj}} = 1 \right\},
\end{aligned}
$$

Identify G_f with the subgroup of diagonal matrices of $\mathrm{GL}(N; \mathbb{C})$.

Remark 10

A pair (f, G), $G \subset G_f$, is often called a *Landau-Ginzburg orbifold*.

Definition 11 (Berglund–Henningson '95)

For a subgroup $G \subset G_f$, define

$$\widetilde{G} := \mathrm{Hom}(G_f/G, \mathbb{C}^*).$$

Berglund–Henningson expect that

$$(f, G) \overset{\text{mirror dual}}{\Longleftrightarrow} (\widetilde{f}, \widetilde{G}).$$

The topological mirror symmetry has been studied by many people including Berglund–Hübsch '93, Kreuzer '94, Berglund–Henningson '95, Kawai–Yang '95, Krawitz '09.

Finally, a simple and readable proof is given by

Theorem 12 (Ebeling–Gusein-Zade–T '16, 8 pages)

We have the following duality of "E-functions"

$$E(f,G)(t^{-1},\overline{t}) = (-1)^n E(\widetilde{f},\widetilde{G})(t,\overline{t}),$$

which yields the duality of "Hodge numbers" $h^{p,q}(f,G) = h^{n-p,q}(\widetilde{f},\widetilde{G})$.

Notations

Definition 13

Set

$$\mathrm{Fix}(g) := \{x \in \mathbb{C}^N \mid g \cdot x = x\}, \quad N_g := \dim_{\mathbb{C}} \mathrm{Fix}(g),$$

$$f^g := f|_{\mathrm{Fix}(g)}.$$

Note that $\mathrm{Fix}(g)$ is a linear subspace of $\mathbb{C}^N$.

Proposition 14

The polynomial f^g is also invertible. For each $g \in G$, we have a natural surjective $\mathbb{C}$-algebra homomorphism $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$ and a natural $\mathrm{Jac}(f)$-module structure on Ω_{f^g}.

Each element $g \in G$ has a unique expression of the form

$$g = \operatorname{diag}\left(\mathbf{e}\left[\frac{a_1}{r}\right], \ldots, \mathbf{e}\left[\frac{a_N}{r}\right]\right)$$

with $0 \leq a_i < r$, where r is the order of g and

$$\mathbf{e}[*] := \exp\left(2\pi\sqrt{-1} \cdot *\right)$$

Definition 15 (Ito–Reid)

The *age* of $g \in G$ is the rational number defined by

$$\operatorname{age}(g) := \frac{1}{r}\sum_{i=1}^{N} a_i.$$

If $G \subset G_f \cap \mathrm{SL}(N; \mathbb{C})$, then $\operatorname{age}(g)$ is an integer.

Definition of $\Omega_{f,G}$

Definition 16

Define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module $\Omega'_{f,G} = \left(\Omega'_{f,G}\right)_{\overline{0}} \oplus \left(\Omega'_{f,G}\right)_{\overline{1}}$ by

$$(\Omega'_{f,G})_{\overline{0}} := \bigoplus_{\substack{g\in G \\ N-N_g \equiv 0 \pmod 2}} \Omega'_{f,g}, \quad (\Omega'_{f,G})_{\overline{1}} := \bigoplus_{\substack{g\in G \\ N-N_g \equiv 1 \pmod 2}} \Omega'_{f,g},$$

$$\Omega'_{f,g} := \Omega_{f^g}.$$

Here, for $g \in G$ with $\mathrm{Fix}(g) = \{0\}$ we define $\Omega_{f^g} := \mathbb{C}1_g$ where 1_g is the constant function 1 on $\{0\}$.

The group G acts on Ω_{f^g} by the pull-back via its action on $\mathrm{Fix}(g)$.

Definition 17

$$\Omega_{f,G} := \left(\Omega'_{f,G}\right)^G.$$

Definition 18 (The orbifold residue pairing)

Define a non-deg. $\mathbb{Z}/2\mathbb{Z}$-graded sym. $\mathbb{C}$-bilinear form on $\Omega'_{f,G}$ by

$$J_{f,G} := \bigoplus_{g\in G} J_{f,g}, \quad J_{f,g} : \Omega'_{f,g} \otimes_{\mathbb{C}} \Omega'_{f,g^{-1}} \longrightarrow \mathbb{C},$$

$$J_{f,g}\left(\omega_g, \omega_{g^{-1}}\right) := \mathbf{e}\left[\frac{1}{2}\mathrm{age}(g^{-1})\right]|G| \cdot \mathrm{Res}_{\mathrm{Fix}(g)}\left[\begin{matrix}\phi_g\phi_{g^{-1}} dx_{i_1}\wedge\cdots\wedge dx_{i_{N_g}} \\ \frac{\partial f^g}{\partial x_{i_1}}\cdots\frac{\partial f^g}{\partial x_{i_{N_g}}}\end{matrix}\right],$$

$$J_{f,g}\left(1_g, 1_{g^{-1}}\right) := \mathbf{e}\left[\frac{1}{2}\mathrm{age}(g^{-1})\right]|G| \quad \text{if } \mathrm{Fix}(g) = \{0\}.$$

Here, $\omega_\bullet = [\phi_\bullet dx_{i_1}\wedge\cdots\wedge dx_{i_{N_g}}] \in \Omega'_{f,\bullet}$, $\bullet = g, g^{-1}$,

Definition 19 (The automorphism group of the pair (f, G))

Define the group $\mathrm{Aut}(f, G)$ of automorphisms of (f, G) as

$$\mathrm{Aut}(f, G) := \{\varphi \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\mathbf{x}]) \mid \varphi(f) = f, \\ \varphi \circ g \circ \varphi^{-1} \in G \text{ for all } g \in G\}.$$

Note that G is naturally identified with a subgroup of $\mathrm{Aut}(f, G)$. The group $\mathrm{Aut}(f, G)$ acts naturally on $\Omega'_{f,G}$ by

$$\Omega'_{f,g} \longrightarrow \Omega'_{f,\varphi\circ g\circ\varphi^{-1}}, \quad \omega \mapsto \varphi^* \omega.$$

Remark 20

We have $\mathrm{Aut}(f, G) = \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\mathbf{x}] * G)$ where $\mathbb{C}[\mathbf{x}] * G$ is the skew group ring.

G-twisted Jacobian algebras

Definition 21

A *G-twisted Jacobian algebra* of f is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra $\mathrm{Jac}'(f, G)$ satisfying the following six axioms:

1. $^{\exists}\mathrm{Jac}'(f,g) \overset{\mathbb{C}\text{-mod}}{\cong} \Omega'_{f,g}$, in particular, $\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f)$ s.t.

$$\mathrm{Jac}'(f, G)_{\overline{0}} := \bigoplus_{\substack{g\in G \\ N-N_g \equiv 0 \pmod 2}} \mathrm{Jac}'(f,g),$$

$$\mathrm{Jac}'(f, G)_{\overline{1}} := \bigoplus_{\substack{g\in G \\ N-N_g \equiv 1 \pmod 2}} \mathrm{Jac}'(f,g).$$

2. The product $\circ$ on $\mathrm{Jac}'(f, G)$ satisfies $\mathrm{Jac}'(f,g) \circ \mathrm{Jac}'(f,h) \subset \mathrm{Jac}'(f,gh)$ and the $\mathbb{C}$-subalgebra $(\mathrm{Jac}'(f,\mathrm{id}), \circ)$ coincides with the $\mathbb{C}$-algebra $\mathrm{Jac}(f)$.

3. The $\mathbb{C}$-module $\Omega'_{f,G}$ has a structure of a $\mathrm{Jac}'(f,G)$-module

$$\vdash : \mathrm{Jac}'(f,G) \otimes \Omega'_{f,G} \longrightarrow \Omega'_{f,G}, \quad X \otimes \omega \mapsto X \vdash \omega,$$

satisfying the following properties:

3.1 For any $g, h \in G$ we have

$$\mathrm{Jac}'(f,g) \vdash \Omega'_{f,h} \subset \Omega'_{f,gh},$$

and the $\mathrm{Jac}'(f,\mathrm{id})$-module structure on $\Omega'_{f,g}$ coincides with the $\mathrm{Jac}(f)$-module structure on Ω_{f^g}.

3.2 By choosing a nowhere vanishing N-form $\widetilde{\zeta}$,

$$\mathrm{Jac}'(f,G) \xrightarrow{\cong} \Omega'_{f,G}, \quad \phi \mapsto \phi \vdash \zeta,$$

where $\zeta := [\widetilde{\zeta}] \in \Omega'_{f,\mathrm{id}} = \Omega_f$.
Namely, $\Omega'_{f,G}$ is a free $\mathrm{Jac}'(f,G)$-module of rank one.

4. $^{\exists}$The induced action of $\mathrm{Aut}(f, G)$ on $\mathrm{Jac}'(f, G)$ given by

$$\varphi^*(X) \vdash \varphi^*(\zeta) := \varphi^*(X \vdash \zeta),\ \varphi \in \mathrm{Aut}(f, G),\ X \in \mathrm{Jac}'(f, G).$$

We require that

$$\varphi^*(X) \circ \varphi^*(Y) = \varphi^*(X \circ Y),\ \varphi \in \mathrm{Aut}(f, G),\ X, Y \in \mathrm{Jac}'(f, G),$$

and it is G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutative, namely, for $g, h \in G$ and $X \in \mathrm{Jac}'(f, g),\ Y \in \mathrm{Jac}'(f, h)$,

$$X \circ Y = (-1)^{\overline{X}\overline{Y}} g^*(Y) \circ X,$$

where g^* is the induced action of $g \in G \subset \mathrm{Aut}(f, G)$.

5. For any $g, h \in G$ and $X \in \mathrm{Jac}'(f, g)$, $\omega \in \Omega'_{f,h}$, $\omega' \in \Omega'_{f,G}$,

$$J_{f,G}(X \vdash \omega, \omega') = (-1)^{\overline{X}\overline{\omega}} J_{f,G}(\omega, (h^{-1})^* X \vdash \omega'),$$

   where $\overline{X}$ and $\overline{\omega}$ are the $\mathbb{Z}/2\mathbb{Z}$-grading of X and ω.

6. Let G' be a subgroup of G_f such that $G \subset G'$.
   Fix a nowhere vanishing N-form $\widetilde{\zeta}$.
   By the axiom 3, the injective map $\Omega'_{f,G} \longrightarrow \Omega'_{f,G'}$ defined by

$$\Omega'_{f,g} \longrightarrow \Omega'_{f,g}, \quad \omega \mapsto \omega,$$

   induces, an injective map of the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-modules $\mathrm{Jac}'(f, G) \to \mathrm{Jac}'(f, G')$, which is an algebra-homomorphism.

Orbifold Jacobian algebras

Theorem 22 (Main Theorem, BTW)

There exists a G-twisted Jacobian algebra $\mathrm{Jac}'(f, G)$ of f unique up to isomorphism.

Lemma 23 (Key Lemma)

The group $G_f \cap \mathrm{SL}(N; \mathbb{C})$ is a proper subgroup of G_f.

Definition 24

The $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra

$$\mathrm{Jac}(f, G) := \left(\mathrm{Jac}'(f, G)\right)^G$$

is called the *orbifold Jacobian algebra* of (f, G).

Historical remarks

Certain works towards the definition of orbifold Frobenius algebras were also done previously by R. Kaufmann and M. Krawitz.

Kaufmann '03 proposed a notion of the *orbifold Frobenius superalgebras ($\mathbb{Z}/2\mathbb{Z}$-algebras)* for (f, G). There one needs to fix a non-unique "choice of a two cocycle".

Krawitz '09 gave a particular construction of an algebra, not a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, for (f, G). However, his "formula" could only be well-defined for weighted homogeneous polynomials. He did not consider the uniqueness of the algebra.

Our definition is valid for all f with finite dimensional $\mathrm{Jac}(f)$. Our axiom 4 on "$Aut(f, G)$-invariance" chooses a particular two cocycle, which reproduces Krawitz's "formula" if $\mathrm{Jac}(f, G) = \mathrm{Jac}(f, G)_{\overline{0}}$ (no odd subspaces).

Product Formula

Let $I_g := \{i_1, \ldots, i_{N_g}\}$ be a subset of $\{1, \ldots, N\}$ such that $\mathrm{Fix}(g) = \{x \in \mathbb{C}^N \mid x_j = 0, j \notin I_g\}$. In particular, $I_{\mathrm{id}} = \{1, \ldots, N\}$. Denote by I_g^c the complement of I_g in I_{id}.

For each $g \in G_f$, there exists $v_g \in \mathrm{Jac}'(f, g)$ such that $\mathrm{Jac}'(f, \mathrm{id}) v_g = \mathrm{Jac}'(f, g)$ and for $g, h \in G_f$ with $I_g^c \cap I_h^c = \emptyset$

$$v_g \circ v_h = \widetilde{\varepsilon}_{g,h} v_{gh}, \quad \widetilde{\varepsilon}_{g,h} \in \{\pm 1\},$$

$$v_g \circ v_{g^{-1}} = (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot [H_{g,g^{-1}}] v_{\mathrm{id}},$$

where

$$H_{g,g^{-1}} := c_g \cdot \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j \in I_g^c}, \quad c_g \in \mathbb{C}^*.$$

Suppose that $f = f(x_1, x_2, x_3)$ defines an ADE singularity.
The holomorphic map $f : \mathbb{C}^3 \longrightarrow \mathbb{C}$ yields $\widehat{f} : \widehat{\mathbb{C}^3/G} \longrightarrow \mathbb{C}$.
We can find a chart $U(\cong \mathbb{C}^3) \subset \widehat{\mathbb{C}^3/G}$ containing all the critical points of $\widehat{f}$. Set $\overline{f} := \widehat{f}|_U$.

Theorem 25 (BTW)

We have an isomorphism of Frobenius algebras

$$\mathrm{Jac}(f, G) \cong \mathrm{Jac}(\overline{f}).$$

Remark 26

W. Ebeling-T show that $\overline{f}$ is an invertible polynomial defining ADE singularities and it has the same "expected geometric properties" as (f, G) does.

Theorem 27 (BTW)

Let $f_1, f_2 \in \mathbb{C}[x, y, z]$ be invertible polynomials defining Arnold's 14 exceptional singularities. There exists an isomorphism of Frobenius algebras

$$\mathrm{Jac}(f_1) \cong \mathrm{Jac}(\widetilde{f_2}, G^{\mathrm{SL}}_{\widetilde{f_2}}), \;\; G^{\mathrm{SL}}_{\widetilde{f_2}} := G_{\widetilde{f_2}} \cap \mathrm{SL}(3; \mathbb{C})$$

if and only if the associated singularities of f_1 and f_2 are strange dual to each other in the sense of Arnold.

Remark 28

The equivalence

$$HMF(f_1) \cong HMF^{G^{\mathrm{SL}}_{\widetilde{f_2}}}(\widetilde{f_2})$$

is given by Carqueville-Ros Camacho-Runkel and Newton-Ros Camacho.

G-equivariant Matrix Factorizations

For $f(\mathbf{x}) = x_1^{a_1} + \cdots + x_n^{a_n}$, we have the $\mathbb{C}$-algebra isomorphism

$$HH^\bullet(MF^G_{\mathbb{C}[\mathbf{x}]}(f)) \cong \mathrm{Jac}(f, G),$$

where we use the following properties:

- $HH^\bullet(MF^G_{\mathbb{C}[\mathbf{x}]}(f)) \cong \bigoplus_{g \in G} \left(HMF_{\mathbb{C}[\mathbf{x},\mathbf{y}]}(f(\mathbf{x}) - f(\mathbf{y}))(\Delta_{\mathrm{id}}, \Delta_g)\right)^G.$
- The product formula of $\mathrm{Jac}'(f, G)$.
- $\mathrm{Jac}'(f_1 + f_2, G_{f_1+f_2}) \cong \mathrm{Jac}'(f_1, G_{f_1}) \otimes \mathrm{Jac}'(f_2, G_{f_2})$.

Want to show this for any invertible polynomial (work in progress).

Thank you very much!