I was not informed before this meeting that my notes would be scanned and put on a CD. Had I known in advance, I would have prepared them with greater care. I apologize for their poor quality.

Mini-course
Cherednik algebras
CIRM, Luminy
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Category $\mathcal{O}$

Reference: V. Ginzburg, N. Guay, E. Opdam, R. Rouquier
math. RT/0212036

W: Weyl group ($\overbrace{\underbrace{\text{finite real reflection group}}_{\text{Coxeter group}}}^{\text{crystallographic}}$) $\mathfrak{h}$: reflection representation of W (over $\mathbb{C}$).

$\mathfrak{h}$: Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$ with associated Weyl group W.

ex: $\mathfrak{h} \cong \mathbb{C}^{n-1}$ = $n \times n$ diagonal matrices of trace 0, $W = S_n$ (type A).

(smash) Semi-direct product: A : ring, G : a finite group acting on A by ring automorphisms $A \rtimes G = \{ ag \mid a \in A, g \in G\}$ $(a_1 g_1)\cdot(a_2 g_2) = (a_1 g_1(a_2))(g_1 g_2)$.

$T(\mathfrak{h} \oplus \mathfrak{h}^*)$ = tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^* = \bigoplus_{k \geq 0} (\mathfrak{h} \oplus \mathfrak{h}^*)^{\otimes k}$

$$\left\{ \begin{array}{lcl} \{\text{reflecting hyperplanes in } \mathfrak{h}\} & \xleftrightarrow{\text{bijection}} & \{\text{reflections in } W\} \\ \{v \in \mathfrak{h} \mid s(v) = v\} & \longleftarrow & s \\ & & \\ V & \longmapsto & \text{pointwise stabilizer of } V \text{ in } W = \{1, s\}. \end{array} \right.$$

Ex: $W = S_n$, $\mathfrak{h} \cong \mathbb{C}^{n-1} \supset \{(d_1, \dots, d_n) \mid \sum_{i=1}^{n} d_i = 0, d_j = d_k\} = \mathfrak{h}_{s_{jk}}$

$\left\{ \begin{array}{l} (\ ,\ ): \text{Killing form, non-degenerate } W\text{-invariant bilinear form on } \mathfrak{h} \\ \text{Ex: } A, B \in \mathfrak{sl}_n.\ (A,B) = \frac{1}{?}\,\mathrm{Tr}(AB). \end{array} \right.$

W is generated by reflections: s.

S = set of reflections in W. For $s \in W$, let $\mathfrak{h}_s = \{v \in \mathfrak{h} \mid s(v) = v\}$ and write $\mathfrak{h} = \mathfrak{h}_s \oplus \mathbb{C}\alpha_s^\vee$: s-stable decomposition $s(\alpha_s^\vee) = -\alpha_s^\vee$. Let $\alpha_s \in \mathfrak{h}^*$ be such that $\alpha_s|_{\mathfrak{h}_s} = 0$ and $\alpha_s(\alpha_s^\vee) = 2$. The α_s's are called **roots**.

$$\{\text{reflections}\} \leftrightarrow \{\text{roots}\}/\{\pm 1\}$$

Ex: $\alpha_{s_{jk}} = \underset{(j<k)}{x_j - x_k}$ $\quad \alpha_{s_{jk}}^\vee = y_j - y_k$ $\quad j<k$.

Let $c : S/W \to \mathbb{C}$ $\quad c$ is a W-invariant function on S.

Fact: There are only two possibilities: $\overset{(W \text{ simple})}{S/W}$ has cardinality 1 or 2.

$\Rightarrow c$ takes 1 or 2 values. $\quad$ ex: $W = S_n$

Let $\langle -, - \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ be the canonical pairing.

Def: Let $t \in \mathbb{C}$, c as above. The rational Cherednik algebra $H_{t,c}(W)$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$ by the following relations:

1) $x_1 \otimes x_2 = x_2 \otimes x_1$, $\quad y_1 \otimes y_2 = y_2 \otimes y_1$ $\quad \forall x_1, x_2 \in \mathfrak{h}^*$ $\forall y_1, y_2 \in \mathfrak{h}$

2) $y \otimes x - x \otimes y = t\langle x, y\rangle - \sum_{s \in S} c_s \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee \rangle s$.

$$H_{t=0,\, c=0}(W) \cong \underbrace{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]}_{\text{polynomials on } \mathfrak{h} \oplus \mathfrak{h}^*} \rtimes W$$

Filtration on $H_{t,c}(W)$: $\deg(x) = 1 = \deg(y)$ $\quad x \in \mathfrak{h}^*, y \in \mathfrak{h}$, $\deg(w) = 0$ $\forall w \in W$

$F_0 = \mathbb{C}[W] \subset F_1 \subset F_2 \subset \cdots$ $\quad F_i$ = **span** of monomials of degree $\leq i$.

This filtration comes from the grading on $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$.

$\mathrm{gr}_{F_\cdot}(H_{t,c}(W)) = \bigoplus_{k\geq 0} F_{k+1}/F_k$. This is a ring.

$$F_{k_1+1}/F_{k_1} \times F_{k_2+1}/F_{k_2} \longrightarrow F_{k_1+k_2+2}/F_{k_1+k_2+1}.$$

$\bar{x}, \bar{y} \in F_1/F_0$ $\quad \bar{y}\cdot\bar{x} - \bar{x}\cdot\bar{y} = \overline{yx - xy} \in F_2/F_1$

$= 0$ in F_2/F_1

because $yx - xy \in \mathbb{C}[W] = F_0 \subset F_1$.

$\Longrightarrow \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\rtimes W \longrightarrow \mathrm{gr}_{F_\cdot}(H_{t,c}(W))$.

Theorem (Etingof-Ginzburg): This map is an isomorphism. (PBW-property)

Corollary: Suppose that $\{x_1, .., x_r\}$ is a basis of $\mathfrak{h}^*$
" " $\{y_1, .., y_r\}$ " " " $\mathfrak{h}$.

Then $\{x_1^{a_1} \cdots x_r^{a_r}\, w\, y_1^{b_1} \cdots y_r^{b_r} \mid a_1, .., a_r, b_1, .., b_r \in \mathbb{Z}_{\geq 0}, w \in W\}$ is a vector space basis of $H_{t,c}(W)$

Corollary: The linear map $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \longrightarrow H_{t,c}(W)$ given by multiplication is an isomorphism.

$\Longrightarrow H_{t,c}(W)$ has a (multiplicative) triangular decomposition.

Analogue: $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ $\quad \mathfrak{n}^+$: upper-triangular matrices
$\mathfrak{h}$: diagonal matrices of trace 0. $\mathfrak{n}^-$: lower " "

The PBW theorem for $U(\mathfrak{sl}_n)$ implies that $U(\mathfrak{sl}_n) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{h}) \otimes_{\mathbb{C}} U(\mathfrak{n}^+)$.

From now on, we will assume that $t=1$. (If $t\neq 0$, $H_{t,c}(W) \cong H_{1,\frac{c}{t}}(W)$.)

Standard modules: Let $\sigma \in \mathrm{Irr}(W)$. We can pull-back σ to a representation of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ via $\mathbb{C}[\mathfrak{h}^*] \rtimes W \to \mathbb{C}[W]$.

$$p(x)\cdot w \mapsto p(0) w$$

Set $\Delta(\sigma) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}^{H_{1,c}(W)} \sigma = H_{1,c}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W} \sigma$: left $H_{1,c}(W)$-module.

PBW corollary $\Longrightarrow$ $\Delta(\sigma) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \sigma$ as left $\mathbb{C}[\mathfrak{h}]$-module.

$H_{t,c}(W)$ is also graded: $\deg(y)=1$ $\forall y \in \mathfrak{h}$ $\deg(x)=-1$ $\forall x \in \mathfrak{h}^*$
$\deg(w)=0$ $\forall w \in W$.

$\Longrightarrow$ $\Delta(\sigma)$ is (negatively) graded, $\Delta(\sigma) = \bigoplus_{j\leq 0} \Delta(\sigma)_j$, $\Delta(\sigma)_0 \cong \sigma$.

Up to a shift, this is an inner grading: choose dual bases $\{x_1,\dots,x_r\}$, $\{y_1,\dots,y_r\}$ of $\mathfrak{h}^*$ and $\mathfrak{h}$. Set $\mathbf{h} = -\frac{1}{2}\sum_{i=1}^{r}(x_i y_i + y_i x_i)$.

Lemma: $[\mathbf{h}, x] = -x$ and $[\mathbf{h}, y] = y$ $\forall x \in \mathfrak{h}^*$, $\forall y \in \mathfrak{h}$

Proof:
$$\begin{aligned}
[\mathbf{h}, x] &= -\frac{1}{2}\sum_{i=1}^{r}\left(x_i[y_i, x] + [y_i, x]x_i\right)\\
&= -\frac{1}{2}\sum_{i=1}^{r}\Big(x_i\big(\langle x, y_i\rangle + \sum_{s\in S} c_s \langle \alpha_s, y_i\rangle\langle x, \alpha_s^\vee\rangle s\big)\\
&\qquad + \big(\langle x, y_i\rangle - \sum_{s\in S} c_s \langle \alpha_s, y_i\rangle\langle x, \alpha_s^\vee\rangle s\big)x_i\Big)\\
&= -x + \frac{1}{2}\sum_{s\in S} c_s \sum_{i=1}^{r} \langle \alpha_s, y_i\rangle\langle x, \alpha_s^\vee\rangle (x_i + s(x_i))s\\
&= -x + \frac{1}{2}\sum_{s\in S} c_s \langle x, \alpha_s^\vee\rangle \underbrace{(\alpha_s + s(\alpha_s))}_{=0} s = -x. \ \square
\end{aligned}$$

y_i acts by 0 on $\Delta(\sigma)_0 \cong \sigma$, so, if $v \in \Delta(\sigma)_0$, $\underline{h}(v) = \frac{1}{2}\sum_{i=1}^{r}(x_i y_i - y_i x_i)(v) =$

$$\frac{-1}{2}\sum_{i=1}^{r}\Big(\langle x_i, y_i\rangle - \sum_{s\in S} c_s \langle \alpha_s, y_i\rangle\langle x_i, \alpha_s^\vee\rangle s\Big)(v) = \underbrace{\frac{-1}{2}\Big(-r + \sum_{s\in S} c_s\langle \alpha_s, \alpha_s^\vee\rangle s\Big)}_{z_c \in Z\Gamma = \text{center of } \mathbb{C}[\Gamma] \text{ because } c \text{ is } W\text{-invariant.}}(v)$$

$$= \chi(c,\sigma)\cdot v$$

Recall: Schur's Lemma $\Rightarrow$ z_c acts on σ by a scalar $\chi(c,\sigma)$.

This observation and the previous lemma $\Rightarrow$ $\underline{h}\cdot v = (\chi(c,\sigma)+j)v$ if $v \in \Delta(\sigma)_j$, $j \leq 0$. The grading on $\Delta(\sigma)$, up to the shift $\chi(c,\sigma)$, is given by $\underline{h}$.

Set $H_c = H_{1,c}(W)$.

Corollary: $\Delta(\sigma)$ has a unique simple quotient (denoted $L(\sigma)$).

Proof: This is a standard argument for Verma modules.
Let J be the sum of all the proper submodules of $\Delta(\sigma)$. Then J must be graded via the action of $\underline{h}$. Moreover, $\Delta(\sigma)_0 \cap J = \{0\}$, since any $0 \neq v \in \Delta(\sigma)_0$ generates $\Delta(\sigma)$, hence $\Delta(\sigma)_0 \cap N = \{0\}$ for any proper submodule $N \subsetneq M$.

$\Rightarrow$ J is the unique proper maximal submodule of $\Delta(\sigma)$.

$$0 \to J \to \Delta(\sigma) \to L(\sigma) \to 0. \quad J = \mathrm{rad}(\Delta(\sigma)) \quad \square$$

Costandard modules $\nabla(\sigma)$

Via $\mathbb{C}[\mathfrak{h}] \rtimes W \longrightarrow \mathbb{C}[W]$, we can pull back σ to a (left) $\mathbb{C}[\mathfrak{h}]\rtimes W$-module..

$$p(x)\cdot w \longmapsto p(0)w$$

$\mathrm{Hom}_{\mathbb{C}[\mathfrak{h}]\rtimes W}(H_{0,c}(W), \sigma)$ is a left $H_c(W)$-module via right multiplication on $H_c(W)$.

Let $\nabla(\sigma)$ be the subspace of $\mathbb{C}[\mathfrak{h}^*]$-nilpotent elements in $\mathrm{Hom}_{\mathbb{C}[\mathfrak{h}]\rtimes W}(H_c(W), \sigma)$.

$\nabla(\sigma) = \mathrm{Homgr}_{\mathbb{C}[\mathfrak{h}]\rtimes W}(H_c(W), \sigma)$ where σ is given degree 0.

↳ space of graded homomorphisms.

$\nabla(\sigma)$ can also be defined as the restricted vector space dual of $\Delta(\sigma)$.

Anti-involution $\zeta : H_{t,c}(W) \longrightarrow H_{t,c}(W)$ where $\theta : \mathfrak{h}^* \to \mathfrak{h}$ is the

$$x \longmapsto \theta(x), \quad y \longmapsto \theta^{-1}(y), \quad w \mapsto w^{-1}$$

W-equivariant isomorphism given by $\langle x, y\rangle = (\theta(x), y)$.

If $M \in \mathrm{mod}_L\text{-}H_{t,c}(W)$, then $M^* \in \mathrm{mod}_R\text{-}H_{t,c}(W)$, but, using ζ, we can turn M^* into a left module. $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$.

Suppose that $M \in \mathrm{mod}_L\text{-}H_c(W)$ is the sum of its $\mathfrak{h}$-generalized eigenspaces:

$M = \bigoplus_{\mu\in\mathbb{C}} M[\mu]$ and $(\mathfrak{h}-\mu)^k m = 0$ if $m \in M[\mu]$, $k \gg 0$. Set $M^\vee = \bigoplus_{\mu\in\mathbb{C}} M[\mu]^* \subset M^*$. "restricted dual"

Then $M^\vee \in \mathrm{mod}_L\text{-}H_c(W)$ (using ζ). $\zeta^* : \mathrm{mod}_R\text{-}H_{t,c}(W) \xrightarrow{\sim} \mathrm{mod}_L\text{-}H_{t,c}(W)$

Skip

$$\begin{cases} \Delta(\sigma)^\vee = \zeta^*\left(\mathrm{Homgr}_{\mathbb{C}}\left(H_c(W) \otimes_{\mathbb{C}[\mathfrak{h}]\rtimes W} \sigma, \mathbb{C}\right)\right) \\ = \zeta^*\left(\mathrm{Homgr}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}\left(H_c(W), \sigma^*\right)\right) \\ = \mathrm{Homgr}_{\mathbb{C}[\mathfrak{h}]\rtimes W}\left(H_c(W), \sigma\right) = \nabla(\sigma) \end{cases}$$

Hom as right $\mathbb{C}[\mathfrak{h}^*]\rtimes W$-modules.

($\sigma \cong \sigma^*$ as W-module. this is not true in general for complex reflection groups.)

Proposition: $\mathrm{Ext}^i_{H_c(W)}(\Delta(\sigma), \nabla(\tau)) = 0$ if $i \geq 1$ or $i = 0$, $\sigma \neq \tau$

Proof: $\mathrm{Ext}^i_{H_c(W)}(\Delta(\sigma), \nabla(\tau)) = \mathrm{Ext}^i_{\mathbb{C}[\mathfrak{h}]\rtimes W}(\mathrm{Res}_{\mathbb{C}[\mathfrak{h}]\rtimes W} \Delta(\sigma), \tau) = \mathrm{Ext}^i_{\mathbb{C}[\mathfrak{h}]\rtimes W}(\mathrm{Ind}_{\mathbb{C}[W]}^{\mathbb{C}[\mathfrak{h}]\rtimes W} \sigma, \tau)$

$= \mathrm{Ext}^i_{\mathbb{C}[W]}(\sigma, \tau) = 0$ if $i \geq 1$ or $i = 0$, $\sigma \neq \tau$. □

(Here, we use that the induction functor is left adjoint to restriction)

$\mathrm{Hom}_A(M_1, \mathrm{Hom}_B(M_2, M_3)) \cong \mathrm{Hom}_B(M_2 \otimes_A M_1, M_3)$. M_1: left A-module, M_3: left B-module, M_2: B-A-bimodule.

$\uparrow$

Insert here the definition of $\mathcal{O}_c$ from the next page.

Definition of block: A block of a finite dimensional algebra is an indecomposable two-sided ideal. A block is also an equivalence class of indecomposable modules, $M \sim N$ if there exists $M_1 = M, M_2, \ldots, M_k = N$, such that M_j and M_{j+1} have a composition factor in common, M_j being also indecomposable.

Prop.: $\{L(\sigma)\}_{\sigma \in \mathrm{Irr}(W)}$ is the set of simple objects in $\mathcal{O}_c$

Proof: Let $M \in \mathcal{O}_c$ be simple and choose $m \in M$ such that $ym = 0$ $\forall y \in \mathfrak{h}$. We can assume that $W.m \cong \sigma \subset M$ for some $\sigma \in \mathrm{Irr}(W)$. Then

$$\begin{aligned} \Delta(\sigma) &\twoheadrightarrow M \\ p &\mapsto pm \end{aligned}$$

is well-defined. $\big(\mathrm{Hom}_{H_c(W)}(\Delta(\sigma), M) \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W}(\sigma, M|_{\mathbb{C}[\mathfrak{h}] \rtimes W}$

$\Rightarrow M \cong L(\sigma)$ $\square$

We can put a partial order $\leq$ on $\mathrm{Irr}(W)$ which turns out to be very important for the representation theory of $H_c(W)$: $\sigma \leq \tau$ if $\sigma \cong \tau$ or $x(c,\tau) \in x(c,\sigma) + \mathbb{R}_{>0}$ (It is possible to have $x(c,\sigma) = x(c,\tau)$ and $\sigma \not\cong \tau$.) We could replace $\mathbb{R}_{>0}$ by $\mathbb{Z}_{>0}$ since $L(\sigma)$ and $L(\tau)$ are not in the same block if $L(\tau) - L(\sigma) \notin \mathbb{Z}$.

Prop.: If $L(\sigma)$ is a subquotient of $\Delta(\tau)$, then $\sigma \leq \tau$.

Proof: Suppose that $M \subset N \subset \Delta(\tau)$ and $N/M \cong L(\sigma)$. Then $\Delta(\sigma) \twoheadrightarrow L(\sigma) \cong N/M \hookrightarrow \Delta(\tau)/M$. $\underline{h}$ acts on a highest weight vector of N/M by $x(c,\sigma)$. Since $\Delta(\tau) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \tau$, all its $\underline{h}$-weights are in $x(c,\tau) - \mathbb{Z}_{\geq 0}$. $\Rightarrow x(c,\sigma) \in x(c,\tau) - \mathbb{Z}_{\geq 0}$ $\square$