I was not informed before this meeting that my notes would be scanned and put on a CD. Had I known in advance, I would have prepared them with greater care. I apologize for their poor quality.
Category 0

Reference: V. Ginzburg, N. Guay, E. Opdam, R. Rouquier
math.RT/0212036

\[ W: \text{Weyl group (finite real reflection group)} \quad b: \text{reflection representation of } W \text{ (over } \mathbb{C}) \]

\[ b: \text{Cartan subalgebra of a simple Lie algebra } \mathfrak{g} \text{ with associated Weyl group } W. \]

ex: \( b \cong \mathbb{C}^{n^2} = \text{n x n diagonal matrices of trace 0} \), \( W = S_n \) (type A).

\[ (\text{smash}) \]

Semi-direct product: \( A: \text{ring}, \ G: \text{a finite group acting on } A \text{ by ring automorphisms} \]

\[ A \rtimes G = \{ a_g | a \in A, g \in G \} \quad (a_g)(a_{g'}) = (a, g)(a, g) \]

\[ T(b \otimes x) = \text{tensor algebra of } b \otimes x = \bigoplus_{k=0}^\infty (b \otimes x)^{\otimes k} \]

\[
\left\{ \begin{array}{c}
\{ \text{reflecting hyperplanes in } b \} \\
\{ V \in GL(V) | s(V) = V \}
\end{array} \right\} \xrightarrow{\text{bijection}} \left\{ \text{reflections in } W \right\}
\]

\[ V \xrightarrow{\text{pointwise stabilizer of } V \text{ in } W = \{1, x\}} \]

Ex: \( W = S_n, \ b = \mathbb{C}^{n^2} = \left\{ (d_1, \ldots, d_n) \mid \sum d_i = 0, d_i d_j = d_j d_i \right\} \)

\[ \left\{ \begin{array}{c}
\{ \text{Killing form, non-degenerate } W\text{-invariant bilinear form on } b \}
\end{array} \right\}
\]

Ex: \( A, B \in \mathfrak{sl}_n \), \( (A, B) = \frac{1}{n} \text{Tr}(AB) \).
\[ S = \text{set of reflections in } W. \text{ For } s \in W, \text{ let } b_s = \{ v \in V \mid s(v) = v \} \] and write 
\[ b = b_s \oplus b_v : s \text{-stable decomposition } s(b_v) = -b_v. \text{ Let } \alpha_s \in b^* \text{ be such that } \alpha_s | b_s = 0 \text{ and } \alpha_s(b_v) = 2. \text{ The } \alpha_s \text{ are called roots.} \]

Ex: \[ \alpha_{x_j - x_k}^* = x_j - x_k \quad \alpha_{x_j - x_k}^* = \frac{x_j - x_k}{jk}. \]

Let \( c : \frac{b^*}{W} \to \mathbb{C} \) is a \( W \)-invariant function on \( S \).

Fact: There are only two possibilities: \( \frac{b^*}{W} \) has cardinality 1 or 2.

\[ \Rightarrow c \text{ takes } 1 \text{ or } 2 \text{ values.} \]

Ex: \( W = S_n \)

Let \( \langle , , \rangle : b^* \times b^* \to \mathbb{C} \) be the canonical pairing.

Def: Let \( t \in \mathbb{C} \) as above. The \( \text{universal Chevalley algebra} \) \( \mathfrak{H}_t(W) \) is the quotient of \( T(b^* \times b^*) \) by the following relations:

1) \( \alpha_x \alpha_y = \alpha_y \alpha_x, \quad \alpha_x \alpha_y \alpha_x = \alpha_y \alpha_x \alpha_y \quad \forall \alpha_x, \alpha_y \in b^* \quad \forall x, y \in b^* \)

2) \( \alpha_x \alpha_y - \alpha_y \alpha_x = t \langle x, y \rangle - \sum_{\alpha_s} c_s \langle \alpha_s, x \rangle \langle y, x \rangle = s \).

\[ \mathfrak{H}_t = \mathbb{C}[b^*] \times W \] 

\[ \text{polynomials on } b^*, \]

Filtration on \( \mathfrak{H}_t(W) \): \( \deg(x) = 1 = 2 \deg(y), x \in b^*, y \in b, \deg(w) = 0 \forall w \in W \)

\[ F_0 = \mathbb{C}W, F_1, F_2, \ldots, F_i = \text{span of monomials of degree } \leq i. \]

This filtration comes from the grading on \( T(b^* \times b^*) \times W \).
\[ \mathfrak{g}^\vee (H^*_c(W)) = \bigoplus_{k \geq 0} \frac{F_{k+1}}{F_k} / \frac{F_k}{F_0} \quad \text{This is a ring.} \]

\[ \frac{F_{k+1}}{F_k} \times \frac{F_{k+1}}{F_k} \longrightarrow \frac{F_{k+b+2}}{F_{k+b+1}} \]

\[ x, y \in \frac{F_k}{F_0}, \quad y \cdot x - x \cdot y = \frac{yx - xy}{F_0} \]

\[ = 0 \quad \text{in } \frac{F_k}{F_0}, \]

because \(yx - xy \in \mathfrak{g}[W] = F_0 \subset F_1.\)

\[ \implies \mathfrak{g}^\vee (H^*_c(W)) \]

**Theorem (Etingof-Ginzburg):** This map is an isomorphism. \((\text{PBW-property})\)

**Corollary:** Suppose that \(\{x_1, \ldots, x_r\}\) is a basis of \(\mathfrak{g}^*\)

\[ \{y_1, \ldots, y_s\} \quad \text{is a basis of } \mathfrak{g} \]

Then \(\{x_1^{a_1} \cdots x_r^{a_r} w_1^{b_1} \cdots w_s^{b_s} | a_i, b_i \in \mathbb{Z}_{\geq 0}, w \in W \}\) is a vector space basis of \(H^*_c(W)\)

**Corollary:** The linear map \(\mathfrak{g}[\mathfrak{b}] \otimes \mathfrak{g}[W] \otimes \mathfrak{g}[\mathfrak{b}] \longrightarrow H^*_c(W)\) given by multiplication is an isomorphism.

\[ \implies H^*_c(W) \text{ has a (multiplicative) triangular decomposition.} \]

**Analogue:** \(\mathfrak{gl}_n = \mathfrak{m} \oplus \mathfrak{n}^+ \oplus \mathfrak{m}^+; \text{ upper-triangular matrices} \)

\(\mathfrak{n}^+: \text{ diagonal matrices of trace 0, } \mathfrak{m}^+: \text{ lower}\)

The PBW theorem for \(U(\mathfrak{g})\) implies that \(U(\mathfrak{gl}_n) = U(\mathfrak{m}) \otimes U(\mathfrak{n}) \otimes U(\mathfrak{m}^+).\)
From now on, we will assume that \( t = 1 \) (if \( \quad \quad H_{t}(W) \equiv H_{1}(W) \)).

Standard modules: Let \( \sigma \in \text{Irr}(W) \). We can pull-back \( \sigma \) to a representation of \( A[\xi] \times W \) via \( A[\xi] \times W \rightarrow A[W] \).

Set \( \Delta(\sigma) = \text{Ind}_{A[\xi] \times W}^{A[W]} H_{t}(W) \). \( \Delta(\sigma) \) is a left \( H_{t}(W) \)-module.

PBW corollary \( \Rightarrow \Delta(\sigma) \cong A[\xi] \otimes \sigma \) as left \( A[\xi] \)-module.

\( H_{t}(W) \) is also graded:

\[
\begin{align*}
\deg(y) &= 1 \quad \forall y \in b^* \\
\deg(w) &= 1 \quad \forall w \in W.
\end{align*}
\]

\( \Rightarrow \Delta(\sigma) \) is (negatively) graded, \( \Delta(\sigma) = \bigoplus_{j \geq 0} \Delta(\sigma)_j \). \( \Delta(\sigma)_0 \cong \sigma \).

Up to a shift, this is an inner grading: choose dual bases \( \{ x_{li} \} \text{ and } \{ y_{li} \} \) of \( b^* \) and \( b \). Set \( \mathfrak{h} = \frac{1}{2} \sum_{i=1}^{r}(x_i y_i + y_i x_i) \).

Lemma: \( [b, x_i] = -x_i \) and \( [b, y_i] = y_i \) \( \forall i \in \mathbb{N} \).

Proof: \( [b, x_i] = -\frac{1}{2} \sum_{i=1}^{r}(x_i [x_i, x_i] + [x_i, x_i] x_i) \)

\[
= -\frac{1}{2} \sum_{i=1}^{r} \left( x_i (x_i x_i + \sum_{s \in S} c_s <x_i y_i> x_i x_i) + (x_i y_i - \sum_{s \in S} c_s <x_i y_i> x_i x_i) x_i \right)
\]

\[
= -x_i + \frac{1}{2} \sum_{s \in S} c_s \sum_{i=1}^{r} <x_i y_i> <x_i y_i> (x_i + s(\mathfrak{h})) s
\]

\[
= -x_i + \frac{1}{2} \sum_{s \in S} c_s <x_i y_i> (\mathfrak{h} + s(\mathfrak{h})) s = -x_i \mathfrak{h}.
\]

\[
= 0
\]
Y acts by D on \( \Delta(\sigma)_0 \equiv \sigma, \sigma_0 \), if \( \forall \in \Delta(\sigma)_0 \), \( h(v) = \frac{1}{2} \sum_{i=1}^{k} \left[ x_i y_i - \sum_{s \in \sigma} \gamma_i < x_i, y_i > x_i, y_i > s \right] (v) \).

\[
\frac{1}{2} \sum_{i=1}^{k} \left[ x_i y_i - \sum_{s \in \sigma} \gamma_i < x_i, y_i > x_i, y_i > s \right] (v) = \frac{1}{3} \left( h + \sum_{s \in \sigma} \gamma_i x_i y_i < x_i, y_i > s \right) (v) \\
\text{where } c \in \mathbb{Z} \Gamma = \text{center of } \mathbb{C}[\Gamma]
\]

Recall: Schur's Lemma \( \Rightarrow h \) acts on \( \sigma \) by a scalar \( x(c, \sigma) \).

This observation and the previous lemma \( \Rightarrow h \cdot v = (x(c, \sigma) + j) v \) if \( \forall \sigma \).

The grading on \( \Delta(\sigma) \), up to the shift \( x(c, \sigma) \), is given by \( h \).

Set \( H_c = H_{\mathbb{C}[\Gamma]}(W) \).

Corollary: \( \Delta(\sigma) \) has a unique simple quotient (denoted \( L(\sigma) \).

This is a standard argument for Verma modules.

Proof: Let \( J \) be the sum of all the proper submodules of \( \Delta(\sigma) \). Then \( J \) must be graded via the action of \( h \). Moreover, \( \Delta(\sigma)_0 \cap J = \{0\} \), since any \( 0 \neq v \in \Delta(\sigma)_0 \) generates \( \Delta(\sigma) \), hence \( \Delta(\sigma)_0 \cap N = \{0\} \) for any proper submodule \( N \subseteq M \).

\( \Rightarrow J \) is the unique proper maximal submodule of \( \Delta(\sigma) \).

\[
0 \rightarrow J \rightarrow \Delta(\sigma) \rightarrow L(\sigma) \rightarrow 0 \quad J = \text{rad}(\Delta(\sigma)) \]

(Standard modules \( V(\sigma) \))

Via \( C[\mathbb{Z}] \times W \rightarrow C[W] \), we can pull back \( \sigma \) to a (left) \( C[\mathbb{Z}] \times W \)-module.

\[
p(\sigma) : W \rightarrow p(\sigma) W
\]

\( \text{Hom}_{C[\mathbb{Z}] \times W}(H_c(W), \sigma) \) is a left \( H_c(W) \)-module via right multiplication on \( H_c(W) \).

Let \( \mathbb{D}(\sigma) \) be the subspace of \( C[\mathbb{Z}] \)-nilpotent elements in \( \text{Hom}_{C[\mathbb{Z}] \times W}(H_c(W), \sigma) \).
\[ \Delta(\sigma) = \text{Hom}_{\mathbb{K}(W)}(H_c(W), \sigma) \text{ where } \sigma \text{ is given degree 0.} \]

space of graded homomorphisms.

\[ \Delta(\sigma) \] can also be defined as the restriction wider space dual of \( \Delta(\sigma) \).

Anti-involution \( S : H_c(W) \to H_c(W) \) where \( \Theta : J^* \to J^* \) is the

\[
\begin{align*}
\Theta(x) &= y, \\
\Theta(y) &= x
\end{align*}
\]

\( \mathbb{K} \)-equivariant isomorphism given by \( \langle x, y \rangle = (x(y), y(x)) \).

If \( M \in \text{mod}_L H_c(W) \), then \( M^* \in \text{mod}_R H_c(W) \), but, using \( S \), we can turn \( M^* \) into a left module.

\[ \text{Hom}_L(M, \sigma). \]

Suppose that \( M \in \text{mod}_L H_c(W) \) is the sum of its \( \mathbb{K} \)-eigenspaces:

\[ M = \bigoplus_{\mu \in \mathbb{C}} H_M^\mu \text{ and } (\lambda - \mu)^{m_\mu} = 0 \text{ if } m \in H_M^\mu. \]

Set \( \sigma^* = \bigoplus_{\mu \in \mathbb{C}} \sigma^*_\mu \subseteq M^* \).

Then \( M^* \in \text{mod}_L H_c(W) \) (using \( S \)).

\[ \Delta(\sigma)^* = S^*(\text{Hom}_{\mathbb{K}(W)}(H_c(W), \sigma) \otimes \mathbb{K}^n, C)) \to \text{Hom as right } \mathbb{K}(W) \text{-modules.} \]

\[ S^*(\text{Hom}_{\mathbb{K}(W)}(H_c(W), \sigma)) = \text{Hom}_{\mathbb{K}(W)}(H_c(W), \sigma) = \Delta(\sigma) \]

Proposition: \( \text{Ext}^i_{H_c(W)}(\Delta(\sigma), \Delta(\tau)) = 0 \) if \( i \geq 1, 1 = 0, \sigma \neq 0 \).

Proof: \( \text{Ext}^i_{H_c(W)}(\Delta(\sigma), \Delta(\tau)) = \text{Ext}^i_{\mathbb{K}(W)}(\text{Res}^i_{\mathbb{K}(W)}(\Delta(\sigma)), \tau) = \text{Ext}^i_{\mathbb{K}(W)}(\sigma, \tau) = 0 \) if \( i \geq 1, 1 = 0, \sigma \neq 0 \).

\[ \text{Hom}_A(M_1, \text{Hom}_B(M_2, M_3)) \cong \text{Hom}_B(M_2 \sigma A M_1, M_3). \]

\( M_1 \) : left \( A \)-module, \( M_2 \) : left \( B \)-module, \( M_3 \) : \( B \)-\( A \)-bimodule.
Insert here the definition of $\Omega$ from the next page.

Definition of block: A block of a finite dimensional algebra is an indecomposable two-sided ideal. Two blocks $M, N$ are in the same block if they have a composition factor in common, $M$ and $N$, being also indecomposable.

Prop: $\{ L(\sigma) \mid \sigma \in \text{Irr}(W) \}$ is the set of simple objects in $\Omega$.

Proof: Let $M \in \Omega$ be simple and choose $m \in M$ such that $\gamma_m = 0$ for all $\gamma \in \text{Irr}(W)$. Then $\Delta(\sigma) = M$ is well-defined. Let $\text{Hom}_{\mathfrak{H}_c}(\Delta(\sigma), M) = \text{Hom}_{\mathfrak{H}_c}(\sigma, M) = \{ m \}$.

$\Rightarrow M \in L(\sigma)$.

We can put a partial order $\preceq$ on $\text{Irr}(W)$ which turns out to be very important for the representation theory of $\mathfrak{H}_c(W)$: $\sigma \preceq \tau$ if $\sigma \leq \tau$ in $\text{Irr}(W)$. It is possible to have $x(c, \sigma) = x(c, \tau)$ and $\sigma \neq \tau$.

We could replace $\mathbb{P}_{\mathbb{Z}_2}$ by $\mathbb{Z}_2$ since $L(\sigma)$ and $L(\tau)$ are not in the same block if $L(\sigma) - L(\tau) \notin \mathbb{Z}_2$.

Prop: If $L(\sigma)$ is a subquotient of $A(\tau)$, then $\sigma \preceq \tau$.

Proof: Suppose that $M \in N \in A(\tau)$ and $N_M \in L(\sigma)$. Then $A(\sigma) \hookrightarrow L(\sigma) \cong N_M \hookrightarrow A(\tau)$. $A$ acts on a highest weight vector of $N_M$ by $x(c, \sigma)$. Since $A(\tau) \cong C[1]$, all its $k$-weights are in $x(c, \tau) - \mathbb{Z}_{20}$. $\Rightarrow x(c, \sigma) \in x(c, \tau) - \mathbb{Z}_{20}$. $\Box$
Corollary: If $c_5 > 0$ for all $c_5$, then $\Delta(\text{sign})$ is simple. ($\text{sign} = \Lambda^1$)

Proof: In this case, $x(c, \text{sign}) = \frac{1}{2} (r + 2 \sum c_5)$. \[x(c, \sigma) \leq \frac{1}{2} \dim \sigma - \sum c_5 \dim \sigma \]

if $\sigma \neq \text{sign}$ because $\sigma$ can only have eigenvalues $1$ or $-1$ on $c_5$, so $\dim c_5 > \dim \sigma$

if $\sigma \neq \text{sign}$

$\Rightarrow x(c, \sigma) > x(c, \text{sign})$, hence $\Lambda(\sigma)$ cannot be a subquotient of $\Delta(\text{sign})$. $\square$

Corollary: $\Delta(\sigma)$ has finite length.

Proof: The $\lambda$-weight spaces of $\Delta(\sigma)$ are all finite dimensional since $\Delta(\sigma) \cong C[\lambda] \otimes \mathcal{O}_C$. If $M \in \Delta(\sigma)$ and $v$ is a vector in $M$ of highest weight, then its weight is $x(c, \sigma)$ for some $c$, so there are only finitely many possibilities. The difference between the $\lambda$-weight of $v$ and $x(c, \sigma)$ cannot be arbitrarily large.

Prop: Any $M \in \mathcal{O}_C$ has finite length.

Def: The category $\mathcal{O}_C$ is the category of finitely generated $\mathcal{O}_C$-modules which are locally nilpotent over $C[\lambda^\ast]$.

If $M \in \mathcal{O}_C$, $M = \bigoplus_{c \in \mathbb{Z}_0} M[c]$ with $M[c] = \{ m \in M \mid (\lambda - c)^m = 0 \}$ for some $k > 0$.

$\dim M[c] < \infty$ for $c \in \mathbb{Z}_0$ large enough and $c \in \mathcal{O}_C(x(c, \sigma) + \mathbb{Z}_0)$

The main structure theorem about $\mathcal{O}_C$ states that it is a highest weight category.

Def: (Clue-Parshall-Scott): A highest weight category $C$ is an abelian, $\mathcal{O}_C$-linear (free field) and cotilting category admitting finite direct sums and having enough projective objects, such that there exists a partial $\Lambda$ satisfying the following conditions:
1. There is a complete collection \( \{ S(\lambda) \} \) of pairwise non-isomorphic simple objects indexed by the set \( \Lambda \).

2. There is a collection \( \{ V(\lambda) \} \), \( \lambda \in \Lambda \), of objects \( C \)-called standard objects such that for each \( \lambda \), an epimorphism \( V(\lambda) \to S(\lambda) \) and all composition factors \( S(\mu) \) \( \mu \neq \lambda \), have finite length \( V_\lambda \in \Lambda \), \( \dim \text{Hom}_C(V(\lambda), V(\mu)) \) and the multiplicities \( [V(\lambda) : S(\mu)] \) are finite.

3. Each simple object \( S(\lambda) \) has a projective cover \( P(\lambda) \) in \( C \). Furthermore, \( P(\lambda) \) has a standard filtration, that is, a decreasing, finite filtration \( P(\lambda) = F^*(\lambda) \subseteq F'(\lambda) \subseteq \cdots \) such that:
   
   (a) \( P(\lambda)/F^*(\lambda) \cong V(\lambda) \)

   (b) \( F^*(\lambda)/F_j(\lambda) \cong V(\mu) \) for some \( \mu \geq \lambda \).

The main example of such a category is the category \( O(\mathfrak{g}) \) for a semi-simple Lie algebra \( \mathfrak{g} \) introduced by Bernstein-Gelfand-Gelfand in the 1970s.

To prove that \( O_c \) is a highest weight category with \( \Lambda = \text{Ind}W \), we have to construct projective modules and show that 3 holds.

- Given \( a \in C \), let \( N(a) \in \mathbb{Z}_{\geq 1} \) such that, if \( \lambda \in \Lambda_c \) and \( m \in \text{mult}(\lambda) \), then \( \lambda, \mu \in \Lambda \).

This is true because there is an upper bound on the possible \( \lambda \)-weights which are in \( a + \mathbb{Z} \). The set of possible \( \lambda \)-weights of modules in \( O \) is \( \bigcup_{a \in C} \{ \lambda \in \mathbb{Z} \} \) for \( a + \mathbb{Z} \).

For \( r \in \mathbb{Z} \geq 1 \), let \( R(a) = H_c(W) \otimes \bigoplus_{a+\mathbb{Z} \in C} \bigoplus_{\lambda \in \Lambda} \bigoplus_{m} \bigoplus_{s \in \mathbb{Z} \geq 1} \bigoplus_{a \in C} \bigoplus_{\lambda \in \Lambda} \mathbb{Z} \).

Then \( R(a) = R(a) \bigoplus (a \otimes \mathbb{Z}) \).

Def: \( O' \) is the full subcategory of \( O \) consisting of modules \( N \) such that, if \( m \in \text{mult}(\lambda) \), then \( (a-\lambda)^m \equiv 0 \).

Ex: \( O'(\sigma), \Omega(\tau) \in O' \).
Fact: \( \exists R_0 \text{ such that } O_c = O_c^{R_0} \). (Proof on the back.)

Prop: \( R(a, R_0) \) is a projective module in \( O_c \). (\( R(a, R) \) is projective in \( O_c^R \)).

Proof: The canonical map \( \hom_{O_c^R}(R(a, R_0), M) \to M[a] \) for \( M \in O_c \) is an isomorphism by the definition of \( R(a, R) \) and \( N(a) \).

Since \( M \to M[a] \) is an exact functor, so is \( \hom_{O_c^R}(R(a, R_0), \cdot) \), hence \( R(a, R_0) \) is projective. □

Prop: \( R(a) \) admits a finite filtration whose successive quotients are standard modules.

Proof: Let \( R_j \) be the \( H_{O_c^R}(w) \)-submodule generated by \( \begin{bmatrix} R(a) \\ C[\theta] \end{bmatrix} \). Then \( R(a) = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_n = \{ 0 \} \).

Then \( R_j / R_{j-1} \cong H_{O_c^R}(w) \otimes_{C[\theta]} \begin{bmatrix} C[\theta] \end{bmatrix} \).

\( \Rightarrow R_j / R_{j-1} \cong \bigoplus_{r \in \text{dimg} W} \Delta(p_j^n(r, j)) \). □

Corollary: \( R(a, R_0) \) also admits such a filtration.

Proof: \( R(a) \to R(a, R_0) \) and \( R(a, R_0) \) is projective, so this epimorphism splits.

Standard fact: if \( M, M_2 \in O \) and \( M_1 \oplus M_2 \) possesses a standard filtration, then so do \( M_1 \) and \( M_2 \). □

Is use the criterion \( \text{Ext}^1_{O_c^R}(\cdot, \Delta(j)) = 0 \) for \( j \in \text{Imm}(\theta) \).
Other argument: In $R(a)$, we have a descending chain of submodules

$$(b-a)M \supset (b-a)^2 M \supset (b-a)^3 M \supset \ldots$$

which must stabilize because $R(a) \in D_c \Rightarrow R(a)$ has finite length. (Due to V. Ginzburg)

Let $R_0 := \max \{R_a\}$ such that $R(a, r) \in R(a, R_0) \forall r > R_0$.

Now let $R_0 = \max \{R_a\}$ such that $x(c, a) \leq x(c, r_0) \forall x(c, a)$ for some $r_0 > R_0$.

If $M \in D_c$, let $\{m_1, \ldots, m_N\}$ be a basis of $\bigoplus_{a \in \mathbb{Q}} H(a)$

Then $\{m_1, \ldots, m_N\}$ generate $M$ and we have an epimorphism

$$\bigoplus_{a \in \mathbb{Q}} R(a, R_0) \twoheadrightarrow M \Rightarrow M \cong \bigoplus_{a \in \mathbb{Q}} R_a.$$
\[ R(x(\zeta, 0), R_0) \rightarrow L(0), \] for some indecomposable direct summand of \( R(x(\zeta, 0), R_0) \) surjects onto \( L(0) \). Call it \( P(0) \). It is projective and it is the projective cover of \( L(0) \) in \( G \). It admits a standard filtration.

We still have to prove that the standard modules in a standard filtration can be ordered in an increasing way. This is essentially a consequence of the following proposition.

Prop. If \( \text{Ext}^1_\mathcal{O}(A(\mu), A(\nu)) \neq 0 \), then \( \mu < \nu \).

Proof. Suppose that \( 0 \rightarrow A(\nu) \rightarrow M \rightarrow A(\mu) \rightarrow 0 \) is an extension of \( A(\mu) \) by \( A(\nu) \). Choose \( x(\zeta, \mu) \in A(\mu) \) and \( \nu \in \text{Ext}^1_\mathcal{O}(u) \cap M[x(\zeta, \mu), \mu] \). Suppose that \( \mu < \nu \) (so that either \( \mu < \nu \) or \( x(\zeta, \mu) - x(\zeta, \nu) \notin Z \)). Then \( x(\zeta, \mu) \) is a maximal \( \mathcal{O}_0 \)-weight of \( M \), so \( y = 0 \). For \( y \), and therefore, \( \Delta = \zeta = x(\zeta, \mu) \).

Let \( N \) be the \( H_{x(\zeta, \mu)} \)-submodule of \( M \) generated by \( m \). Then
\[ \begin{align*}
A(\mu) & \rightarrow N \\
\text{Hom}(A(\mu), x(\zeta, \mu)) & \rightarrow 0. \end{align*} \]
The composite must be an isomorphism, so \( N \cong A(\mu) \) and \( x(\zeta, \mu) \) splits, so the extension is trivial.

**BGG reciprocity:**

**Reminder:** \( \text{Hom}(A(\mu), x(\zeta, \mu)) \neq 0 \) \( \Rightarrow \mu = p \) and \( \text{Ext}^1_\mathcal{O}(A(\mu), x(\zeta, \mu)) = 0 \) \( \forall p \neq p \).

\( \Rightarrow \) The multiplicity \( [P(0): A(\mu)] \) of \( A(\mu) \) in a standard filtration of \( P(0) \) is well-defined and independent of the filtration. More precisely, \( [P(0): A(\mu)] = \dim \text{Hom}(P(0), A(\mu)) \) (use the long exact sequence for Ext).

\[ [P(0): A(\mu)] = [\Delta(\mu): L(0)] \]
\[ = [\Delta(\mu): L(0)] \] since \( G \) has a density such that \( \nabla(\mu) = \Delta(\mu) \).

\[ L(0) \cong L(0). \]