Mini-course
Cherednik algebras
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Finite dimensional representations of rational Cherednik algebras of type A_{n-1} when $t\neq 0$ $(t=1)$

References: math.RT/0208138 Y. Berest, P. Etingof, V. Ginzburg
math.RT/0208126 I. Gordon

Type A_{n-1}

$W=S_n$ $\quad \mathfrak{h}\cong\mathbb{C}^{n-1}$ $\quad y\in\mathfrak{h},\ x\in\mathfrak{h}^*$ $\quad s_{ij}\in S_n$: transposition

$\mathfrak{h}\cong \text{span}\{y_i-y_{i+1} \mid 1\le i\le n-1\}$

It may be more convenient to use the fundamental weights as a basis of $\mathfrak{h}^*$.

$\mathfrak{h}^*\cong\text{span}\{x_i-x_{i+1} \mid 1\le i\le n-1\} \xrightarrow{\sim} \text{span}\{x_1,\dots,x_n\}/(x_1+\dots+x_n=0)$

$\langle\ ,\ \rangle : \mathfrak{h}^*\oplus\mathfrak{h}\to\mathbb{C}$

In $H_{t=1,c}(S_n)$ (H_c), $\quad yx-xy=\langle x,y\rangle - c\sum_{1\le i<j\le n}\langle y,x_i-x_j\rangle\langle y_i-y_j,x\rangle s_{ij}$.

Theorem: i) If $H_{t=1,c}(S_n)$ admits finite dimensional representations, then $c=\pm\frac{r}{n}$ with $r\in\mathbb{Z}_{\ge 1}$, $(r,n)=1$.

ii) If $c=\frac{r}{n}>0$, $(r,n)=1$, the only irreducible finite dimensional representation of $H_{t=1,c}(S_n)$ is $L(\text{triv})$.

$H_{t,c}(W)\cong H_{t,-c}(W)$

If $c=-\frac{r}{n}<0$, $(r,n)=1$, the only irreducible finite dimensional representation of $H_{t=1,c}(S_n)$ is $L(\text{sign})$.

Recall that $\text{Ext}^1_{\mathcal{O}}(L(\tau),L(\tau))=0$. This implies that, if $c=\pm\frac{r}{n}$, $(r,n)=1$, any finite dimensional representation of $H_c(S_n)$ is isomorphic to $L(\text{triv})^{\oplus k}$ or $L(\text{sign})^{\oplus k}$ for some $k\ge 1$, depending on the sign of c.

When does H_c have a one-dimensional representation $V = \mathbb{C} v$?

$yv = 0 = xv$, $\sigma v = v$.

$$0 = (yx - xy)v = \left(\langle y, x\rangle - c \sum_{1 \le i < j \le n} \langle y, x_i - x_j\rangle \langle y_i - y_j, x\rangle \right) v \quad \forall x \in \mathfrak{h}^*, \forall y \in \mathfrak{h}.$$

$$\Leftrightarrow \quad 0 = \left(\langle y, x\rangle - c \cdot h \cdot \langle y, x\rangle \right) v \Leftrightarrow c = \frac{1}{h}$$

h is the Coxeter number of S_n.

Reminder on the Koszul complex: R: commutative ring

$r_1, \dots, r_k$ a sequence of elements in R.

The Koszul complex associated to this data is: $R^k = R^{\oplus k}$: free R-module.

$$0 \to \Lambda^k R^k \to \Lambda^{k-1} R^k \to \dots \to \Lambda^p R^k \xrightarrow{\partial_p} \dots \to \Lambda^1 R^k \to \Lambda^0 R^k \to 0$$

$$\Lambda^0 R^k = R$$

and ∂_p is given by:

$$\partial_p (e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} r_{i_j} e_{i_1} \wedge \dots \wedge \underset{\text{not there}}{\hat{e}_{i_j}} \wedge \dots \wedge e_{i_p} \qquad 1 \le i_1 < \dots < i_p \le k.$$

$e_1, \dots, e_k$ is a basis of the free R-module $R^k = \underbrace{R \oplus \dots \oplus R}_{k \text{ times}}$

Theorem: If $r_1, \dots, r_k$ is a regular sequence in R ($\Leftrightarrow r_j$ is not a zero divisor in $R/(r_1, \dots, r_{j-1})R$), then the Koszul complex is a free resolution of $R/(r_1, \dots, r_k)R$.

Take $R = \mathbb{C}[\mathfrak{h}]$, $r_i = x_i - x_{i+1}$, $1 \le i \le n-1$, so $R/(r_1, \dots, r_{n-1})R \cong \mathbb{C}$. The Koszul resolution of $\mathbb{C}$ is:

$$0 \to \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \Lambda^{n-1}\mathfrak{h}^* \to \dots \to \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \Lambda^2\mathfrak{h}^* \to \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathfrak{h}^* \to \mathbb{C}[\mathfrak{h}] \to \mathbb{C} \to 0.$$

Proposition: Suppose that $c = \frac{1}{n}$. Then the Koszul complex gives a resolution of the one-dimensional $\mathbf{H}_c$-module.

$$0 \to \underset{\Delta(\text{sign})}{\overset{\|}{\Delta(\Lambda^{n-1}\mathfrak{h}^*)}} \to \dots \to \Delta(\Lambda^2\mathfrak{h}^*) \xrightarrow{\partial_2} \Delta(\mathfrak{h}^*) \xrightarrow{\partial_1} \Delta(\text{triv}) \xrightarrow{\partial_0} \overset{\overset{\mathbb{C}}{\|}}{L(\text{triv})} \to 0$$

Proof: We know that $\overset{\overset{\mathbb{C}[\mathfrak{h}]}{\wr\!\wr}}{\Delta(\text{triv})} \twoheadrightarrow L(\text{triv}) \cong \mathbb{C}$ and the kernel of this map is $\mathbb{C}[\mathfrak{h}]_+$, which is generated by $\mathfrak{h}^*$, so $\Delta(\mathfrak{h}^*) \xrightarrow{\partial} \Delta(\text{triv}) \to L(\text{triv}) \to 0$ is exact. (and ∂ is a map of H_c-mod: $\mathfrak{h}^*$ is a space of hwv. of $\Delta(\mathfrak{h}^*)$.) $\mathrm{Ker}(\partial_1)$ is an $\mathbf{H}_c(S_n)$-submodule. From the Koszul resolution of $\mathbb{C}$, we know that it is an epimorphic image of $\mathbb{C}[\mathfrak{h}] \otimes \Lambda^2\mathfrak{h}^*$ as $\mathbb{C}[\mathfrak{h}] \rtimes W$-module. It follows that $\Lambda^2\mathfrak{h}^* \hookrightarrow \mathrm{Im}(\partial_2) = \mathrm{Ker}(\partial_1) \hookrightarrow \Delta(\mathfrak{h}^*)$ is a space of highest weight vectors, so $\Delta(\Lambda^2\mathfrak{h}^*) \underset{\text{is } H_c\text{-map.}}{\overset{\partial_2}{\twoheadrightarrow}} \mathrm{Ker}(\partial_1)$.

Repeating this argument, we conclude that the Koszul complex for $\mathbb{C}$ is a resolution of $L(\text{triv}) \cong \mathbb{C}$ in the category of $\mathbf{H}_{\frac{1}{n}}(S_n)$-modules $\square$.

Theorem: Suppose that $c = \frac{r}{n} > 0$, $(r,n) = 1$, $r \in \mathbb{Z}_{\geq 1}$. $L(\text{triv})$ admits a resolution similar to the Koszul resolution above in the case $c = \frac{1}{n}$ (with $\Delta(\Lambda^k\mathfrak{h}^*)$ instead of $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \Lambda^k\mathfrak{h}^*$).

To prove this theorem and the classification theorem, we need to use the KZ functor and the representation theory of finite Hecke algebras at roots of unity.

$q = e^{2\pi i c}$

$$\mathcal{H}_q(S_n) = \langle T_1, \dots, T_{n-1} \rangle \Big/ \left((T_i - 1)(T_i + q),\ T_i T_{i+1} T_i - T_{i+1} T_i T_{i+1},\ T_i T_j - T_j T_i \text{ if } |i-j|>1 \right)$$

Representation theory of $\mathcal{H}_q(S_n)$: Let e = order of q. ($e = \infty$ if q is not a root of unity). $\lambda \vdash n$: $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots > 0)$ is a partition of n.

Dipper-James, Mathas' book

S^λ: Specht modules of $\mathcal{H}_q(S_n)$. These can be constructed explicitly as submodules of $\mathcal{H}_q(S_n)$. (**Dipper-James**) $\dim(S^\lambda)$ is the number of standard λ-tableaux ($\Rightarrow$ It is independent of q.)

$S^\lambda \twoheadrightarrow D^\lambda$: D^λ is 0 or a simple $\mathcal{H}_q(S_n)$ module.

$\{D^\lambda \mid \lambda \text{ is } e\text{-regular}\}$ = set of all irreducible representations of $\mathcal{H}_q(S_n)$

$\lambda_i - \lambda_{i+1} < e\ \forall i$, (also called e-restricted)

$D^\lambda \neq \{0\} \Leftrightarrow \lambda$ is e-regular

If $\lambda \neq \mu$ and $D^\mu \neq \{0\}$, then $[S^\lambda : D^\mu] > 0 \Rightarrow \mu \lhd \lambda$. $\mu \trianglelefteq \lambda$: $\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_j$

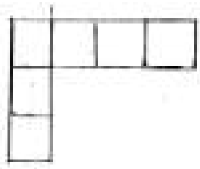
$[S^\lambda : D^\lambda] = 1$ if $D^\lambda \neq \{0\}$.

Definition: λ **is an** e-core if the Young diagram associated to λ contains no hook of length e.

☆ see back

Proposition [DJ]: S^λ and S^μ are in the same block $\Leftrightarrow$ λ and μ have **the same** e**-core**. (this was called the Nakayama conjecture) In particular, **if** λ has no hook of length e, then S^λ is the only Specht module in its block, $D^\lambda \neq \{0\}$ and, actually, $S^\lambda = D^\lambda$.

Suppose now that $q = e^{2\pi i r/n}$, $(r,n) = 1$, so q is a primitive n^{th}-root of unity and $e = n$.

n-core diagrams ∌ single hook diagrams of length n = 

$\Updownarrow$

representations $\Lambda^i \mathfrak{h}^*$ de S_n, $i = n - \lambda_1$

$\lambda^i = (\lambda^i_1 \geq \lambda^i_2 = 1, \lambda^i_3 = 1, \dots)$

Explain here how to compute $\chi(c, \Lambda^i \mathfrak{h}^*)$: see verso

These all have the same n-core. Furthermore, if $\lambda \neq \lambda^i$, then λ has no hook of length n, so $S^\lambda = D^\lambda$ and S^λ is projective (In this case, λ is itself an n-core)

$$\mathcal{H}_q = \bigoplus_{\lambda \text{ is an } n\text{-core}} \mathrm{End}_{\mathbb{C}} S^\lambda \oplus A \quad \text{: block decomposition}$$

where A is an indecomposable, non-semisimple algebra ↳ is indecomposable two-sided ideal

Proposition [DJ]: $D^{\lambda^i} \neq 0 \iff \lambda^i$ is n-regular, $\iff i \neq 0$, $\overbrace{\exists}^{\text{an exact sequence}}$ $0 \to D^{\lambda^{i+1}} \to S^{\lambda^i} \to D^{\lambda^i} \to 0$

Theorem (R. Rouquier): $KZ(\Delta(\sigma)) \cong S^\sigma$ (This is valid for any value of $c \geq 0$.)
(Also isomorphic to a cell module – Kazhdan–Lusztig theory.)

Construction of the BGG resolution: suppose now that $c = \frac{r}{n} > 0$, $(r,n)=1$

We want to extend $\Delta(\mathrm{triv}) \longrightarrow L(\mathrm{triv}) \longrightarrow 0$.

Prop: When $c = \frac{r}{n} > 0$, $(r,n)=1$, $\Delta(\sigma)$ is either simple or has only two composition factors.

Lemma (Opdam–Rouquier): $\mathrm{Hom}_{H_{t=1,c}(S_n)}(\Delta(\sigma),\Delta(\tau)) \longrightarrow \mathrm{Hom}_{\mathcal{H}_q}(KZ(\Delta(\sigma)), KZ(\Delta(\tau)))$ is injective.

$\qquad\qquad\qquad\qquad = \mathrm{Hom}_{\mathcal{H}_q}(S^\sigma, S^\tau)$

$\Rightarrow \mathrm{Hom}_{H_{t=1,c=\frac{r}{n}}}(\Delta(\sigma),\Delta(\tau)) = 0$ if $\tau \neq \Lambda^i\mathfrak{h}^*$ and $\sigma \neq \tau$ since S^τ is then simple. $\Rightarrow \Delta(\tau)$ is simple.

Computation of $\kappa(\varsigma\, \Lambda^i \mathfrak{z}^*)$: $\mathrm{Tr}\left(\sigma_{jk}|_{\Lambda^i\mathfrak{z}^*}\right) = \binom{n-2}{i} - \binom{n-2}{i-1}$

$$= \frac{(n-2)!}{(i-1)!\,(n-2-i)!}\cdot\left(\frac{1}{i} - \frac{1}{n-1-i}\right)$$

$$\kappa(\varsigma, \Lambda^i\mathfrak{z}^*) = \frac{1}{\dim(\Lambda^i\mathfrak{z}^*)}\,\mathrm{Tr}\left(-\frac{n-1}{2} + c\sum_{j<k}\sigma_{jk}\right)\Big|_{\Lambda^i\mathfrak{z}^*}$$

$$= -\frac{(n-1)}{2} + c\cdot\frac{1}{\binom{n-1}{i}}\cdot\binom{n}{2}\,\mathrm{Tr}\left(\sigma_{12}\right)\Big|_{\Lambda^i\mathfrak{z}^*}$$

$$= -\frac{(n-1)}{2} + c\cdot\frac{i!\,(n-1-i)!}{(n-1)!}\cdot\frac{n(n-1)}{2}\cdot\frac{(n-2)!}{(i-1)!\,(n-2-i)!}\,\frac{(n-1-2i)}{i\cdot(n-1-i)}$$

$$= -\frac{(n-1)}{2} + c\,\frac{n}{2}\,(n-1-2i)$$

$$= -\frac{(n-1)}{2} + cn\left(\frac{n-1}{2} - i\right)$$

Proposition: If $\mathrm{Hom}_{H_{\frac{c}{h}}}(\Delta(\sigma), \Delta(\tau)) \neq 0$ and $\sigma \neq \tau$, then $\sigma = \Lambda^{i+1}\mathfrak{h}^*$, $\tau = \Lambda^i \mathfrak{h}^*$
Furthermore, $\dim \mathrm{Hom}_{H_{\frac{c}{h}}}(\Delta(\Lambda^{i+1}\mathfrak{h}^*), \Delta(\Lambda^i \mathfrak{h}^*)) = 1$

Proof: By the Opdam-Rouquier's lemma, $\mathrm{Hom}_{H_{(1), c=\frac{c}{h}}}(\Delta(\sigma), \Delta(\tau)) \neq 0$
$\Rightarrow \mathrm{Hom}_{\mathcal{H}_q}(KZ(\Delta(\sigma)), KZ(\Delta(\tau))) \neq 0 \Rightarrow \mathrm{Hom}_{\mathcal{H}_q}(S^\sigma, S^\tau) \neq 0$.
$\Rightarrow \sigma = \Lambda^{i+1}\mathfrak{h}^*$; $\tau = \Lambda^i \mathfrak{h}^*$. Suppose for the moment that $i \geq 1$.

$$0 \to D^{\lambda^{i+1}} \to S^{\lambda^i} \to D^{\lambda^i} \to 0$$

$$\mathrm{Hom}_{\mathcal{H}_q}(S^{\lambda^{i+1}}, S^{\lambda^i}) = 1 \Rightarrow \mathrm{Hom}_{\mathcal{H}_q}(KZ(\Delta(\Lambda^{i+1}\mathfrak{h}^*)), KZ(\Delta(\Lambda^i \mathfrak{h}^*))) = 1.$$

Opdam-Rouquier

$$\mathrm{Hom}_{H_{\frac{c}{h}}}(\Delta(\Lambda^{i+1}\mathfrak{h}^*), \Delta(\Lambda^i \mathfrak{h}^*))$$

State the Riemann-Hilbert correspondence.

$\Rightarrow \mathrm{Hom}_{\mathcal{D}(\mathfrak{h}^{reg}) \rtimes S_n}(\Delta(\Lambda^{i+1}\mathfrak{h}^*)|_{\mathfrak{h}^{reg}}, \Delta(\Lambda^i \mathfrak{h}^*)|_{\mathfrak{h}^{reg}}) = 1$

Set $M = \Delta(\Lambda^i \mathfrak{h}^*) \cap \mathrm{Image}(\Delta(\Lambda^{i+1}\mathfrak{h}^*)|_{\mathfrak{h}^{reg}})$. (Recall that $\Delta(\sigma) \hookrightarrow \Delta(\sigma)|_{\mathfrak{h}^{reg}}$ since $\Delta(\sigma)$ is a free $\mathbb{C}[\mathfrak{h}]$-module.)

$M \neq \{0\}$ and $M \neq \Delta(\Lambda^i \mathfrak{h}^*)$, for $M = \Delta(\Lambda^i \mathfrak{h}^*) \Rightarrow \Delta(\Lambda^{i+1}\mathfrak{h}^*)|_{\mathfrak{h}^{reg}} \twoheadrightarrow \Delta(\Lambda^i \mathfrak{h}^*)|_{\mathfrak{h}^{reg}} \Rightarrow KZ(\Delta(\Lambda^{i+1}\mathfrak{h}^*)) \twoheadrightarrow KZ(\Delta(\Lambda^i \mathfrak{h}^*))$

$\Rightarrow S^{\lambda^{i+1}} \twoheadrightarrow S^{\lambda^i}$, which is not true (since $i \geq 1$).

$\Rightarrow \{0\} \subsetneq M \subsetneq \Delta(\Lambda^i \mathfrak{h}^*)$.

The Riemann-Hilbert correspondence (Deligne's book on systems of differential equations with regular singularities

X: smooth complex algebraic variety

Theorem: There is an equivalence of categories between

1) The category of algebraic vector bundles on X with a regular, integrable connection.

2) ——— holomorphic ——— with an integrable connection on X^{an}

3) Local systems (locally constant sheaves) on X^{an}

monodromy 4) Complex representations of finite dimension of the fundamental group $\pi_1(X^{an}, x_0)$.

The equivalence 1) $\Leftrightarrow$ 2) requires a proof because the results of GAGA are valid only for projective varieties.

$\underbrace{\text{GAGA}}_{\text{on equivalence of categories of sheaves}}$