Mini-course
Cherednik algebras
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Finite dimensional representations of rational Cherednik algebras for Weyl groups when $t \neq 0$ ($t=1$).

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W: Weyl group $W \circlearrowright \mathfrak{h}$

Let h be the Coxeter number of W, $h = \dfrac{2|R_+|}{\dim \mathfrak{h}}$

↳ See the definition on the back

Goal: Study $\mathcal{O}(H_{t=1,c}(W))$ when $c = \frac{1}{h} + m$, $m \in \mathbb{Z}_{\geq 0}$, in particular finite dimensional modules.

(so c is constant, $c_{long} = c_{short}$)

Easy case: When $c = \frac{1}{h}$, $H_c(W)$ has a one-dimensional module.

Main ideas: KZ functor + $\mathcal{H}_q$..., use the (anti-) spherical subalgebra and shift functors. The Koszul resolution is not necessary, although it exists.

Lemma: Suppose $c = \frac{1}{h} + m$, $m \in \mathbb{Z}_{\geq 1}$. Let $\tau, \mu \in \mathrm{Irr}(W)$, $\tau \neq \mu$ and $\mathrm{Hom}_{H_{t=1,c}(W)}(\Delta(\tau), \Delta(\mu)) \neq 0$; then $\tau \cong \Lambda^i \mathfrak{h}$ and $\mu \cong \Lambda^j \mathfrak{h}$ for some $0 \leq i \neq j \leq \dim \mathfrak{h}$.

Proof: It is similar to the one in Type A. The representation theory of $\mathcal{H}_q(W)$ for $q = e^{2\pi i c}$ implies, via the KZ functor, that $\Delta(\rho)$ is irreducible (and projective) if $\rho \not\cong \Lambda^i \mathfrak{h}$ $\forall\, 0 \leq i \leq \dim \mathfrak{h}$. Furthermore, the $\mathcal{H}_q(W)$-modules $KZ(\Delta(\Lambda^i \mathfrak{h}))$ are all in the same block.

Coxeter elements: If $s_1, \dots, s_n$ are simple reflections in W, then $s_1 s_2 \cdots s_n$ is called a Coxeter element. (These are all conjugate.)
For S_n, the cycle $(1, 2, \dots, n)$ is a Coxeter element.
Def.: The Coxeter number h of W is the order of a Coxeter element.
The Coxeter number of S_n is n.

Lemma: $h = \dfrac{2|R_+|}{\dim \mathfrak{h}}$ $\quad |R_+|$ = number of positive roots

Finally, one uses the following result of E. Opdam and R. Rouquier:

$$\mathrm{Hom}_{H_c}(\Delta(\tau), \Delta(\mu)) \hookrightarrow \mathrm{Hom}_{\mathcal{H}_q(W)}(KZ(\Delta(\tau)), KZ(\Delta(\mu))) \quad \square$$

Let $e = \frac{1}{|W|} \sum_{w\in W} w$, $e_\varepsilon = \frac{1}{|W|} \sum_{w\in W} \mathrm{sign}(w) w$

If $M \in H_{t,c}(W)$-module, $e_\varepsilon M$ is the direct sum of all copies of the sign representation in M.

The eigenvalues of a Coxeter element (acting on $\mathfrak{h}$) are ζ^{e_i}, ζ: primitive h-th root of unity.

$n = \dim \mathfrak{h}$, $e_1 = 1$, $e_n = h-1$

$\sum_{k=1}^{n} e_k = |R_+|$

$d_1 \cdots d_n = |W|$

Lemma: Suppose that $c = \frac{1}{h} + m$. Let $e_1, \ldots, e_n$ be the exponents of W and $d_i = e_i + 1$. (The d_i's are the degrees of the fundamental invariants for W.) Let $p(t) = \prod_{i=1}^{n} \frac{1}{(1-t^{d_i})}$. ($p(t)$ is the Hilbert series of the ring $\mathbb{C}[\mathfrak{h}]^W$.)

The Hilbert series of $e_\varepsilon \Delta(\Lambda^{n-i}\mathfrak{h})$ with respect to the $\mathbf{h}$-eigenspaces.

$$\mathrm{Tr}_{e_\varepsilon \Delta(\Lambda^{n-i}\mathfrak{h})}(t^{\mathbf{h}}) = t^{-m|R_+| + (n-i)(mh+1)} \, p(t) \sum_{1 \le j_1 < \cdots < j_i \le n} t^{e_{j_1} + \cdots + e_{j_i}}.$$

Proof: $\mathrm{Hom}_W(\Lambda^n\mathfrak{h}, \Delta(\Lambda^{n-i}\mathfrak{h})) \cong \mathrm{Hom}_W(\Lambda^n\mathfrak{h}, \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \Lambda^{n-i}\mathfrak{h})$

$\cong \mathrm{Hom}_W(\Lambda^n\mathfrak{h} \otimes (\Lambda^{n-i}\mathfrak{h})^*, \mathbb{C}[\mathfrak{h}])$

$\cong \mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}])$

because $\Lambda^i\mathfrak{h} \otimes_{\mathbb{C}} \Lambda^{n-i}\mathfrak{h} \longrightarrow \Lambda^n\mathfrak{h} \Rightarrow \Lambda^i \cong (\Lambda^{n-i}\mathfrak{h})^* \otimes \Lambda^n\mathfrak{h}$

$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W$ where $\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W} \cong \mathbb{C}[\mathfrak{h}] / (\mathbb{C}[\mathfrak{h}]^W_+)$.

(W-invariant decomposition)

$\mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}]) \cong \mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W}) \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W$

Finally, one uses the following result of E. Opdam and R. Rouquier:

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If $M \in H_{t,c}(W)$-module, $e_\epsilon M$ **is** the direct sum of all copies of the sign representation in M.

The eigenvalues of a Coxeter element (acting on $\mathfrak{h}$) are ζ^{e_i}, ζ: primitive hth root of unity.

$n = \dim \mathfrak{h}$ ($e_1 = 1$, $e_n = h-1$)

$\sum_{k=1}^{n} e_k = |R_+|$ Lemma: Suppose that $c = \frac{1}{h} + m$. Let $e_1, \dots, e_n$ be the exponents of W **and** $d_i = e_i + 1$. (The d_i's are the degrees of the fundamental invariants

$d_1 \cdots d_n = |W|$ for W.) Let $p(t) = \prod_{i=1}^{n} \frac{1}{(1-t^{d_i})}$. ($p(t)$ is the Hilbert series of the ring $\mathbb{C}[\mathfrak{h}]^W$.)

the Hilbert series of $e_\epsilon \Delta(\Lambda^{n-i}\mathfrak{h})$ with respect to the $\mathbf{h}$-eigenspaces.

$$\mathrm{Tr}_{e_\epsilon \Delta(\Lambda^{n-i}\mathfrak{h})}(t^{\mathbf{h}}) = t^{-m|R_+| + (n-i)(mh+1)}\, p(t) \sum_{1 \le j_1 < \dots < j_i \le n} t^{e_{j_1} + \dots + e_{j_i}}$$

Proof:
$$\mathrm{Hom}_W(\Lambda^n\mathfrak{h}, \Delta(\Lambda^{n-i}\mathfrak{h})) \cong \mathrm{Hom}_W(\Lambda^n\mathfrak{h}, \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \Lambda^{n-i}\mathfrak{h})$$
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$$\cong \mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}])$$

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$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W$ where $\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W} \cong \mathbb{C}[\mathfrak{h}] / (\mathbb{C}[\mathfrak{h}]^W_+)$.
(W-invariant decomposition)

$$\mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}]) \cong \mathrm{Hom}_W(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}W}) \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W$$

It is known that $\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}\,W} \cong \mathbb{C}[W]$ as W-modules.

$\Rightarrow \dim_W\left(\Lambda^i\mathfrak{h}, \mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}\,W}\right) = \dim_{\mathbb{C}} \Lambda^i\mathfrak{h} = \binom{n}{i}$.

Fact: There is a copy of $\mathfrak{h}$ in degree e_i $\forall\, 1\leq i\leq n$. Denote it $\mathfrak{h}_i$.

If $1\leq j_1<\dots<j_i\leq n$, $\Lambda^i\mathfrak{h}\cong \mathfrak{h}_{j_1}\wedge\mathfrak{h}_{j_2}\wedge\dots\wedge\mathfrak{h}_{j_i}$ is in degree $e_{j_1}+\dots+e_{j_i}$. Since there are $\binom{n}{i}$ such choices of $1\leq j_1<\dots<j_i\leq n$, we have thus obtained all the copies of $\Lambda^i\mathfrak{h}$ in $\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}\,W}$. □

Very important lemma:

Lemma: Suppose $c=\frac{1}{h}+m$, $m\in\mathbb{Z}_{\geq 1}$. Then $e_\epsilon L(\Lambda^i\mathfrak{h})\neq 0$.

Proof: Concretely this means that sign appears in $L(\Lambda^i\mathfrak{h})$. We have to show that $\mathrm{rad}(\Delta(\Lambda^i\mathfrak{h}))$ does not contain all the copies of sign in $\Delta(\Lambda^i\mathfrak{h})$. We know already that the composition factors of $\mathrm{rad}(\Delta(\Lambda^i\mathfrak{h}))$ are contain $L(\Lambda^j\mathfrak{h})$ for $j>i$. The difference between the lowest degrees of $\mathrm{Tr}_{e_\epsilon\Delta(\Lambda^i\mathfrak{h})}(t^h)$ and $\mathrm{Tr}_{e_\epsilon\Delta(\Lambda^j\mathfrak{h})}(t^h)$ is $mh+1-e_k$ for some k and $mh+1-e_k>0$ since $m\geq 1$ and $e_k<h$ $\forall\, 1\leq k\leq n$. ($e_n=h-1$) This proves our claim: the copy of $\epsilon=\Lambda^n\mathfrak{h}$ of highest degree in $\Delta(\Lambda^i\mathfrak{h})\cong\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}}\Lambda^i\mathfrak{h}$ is not contained in $\mathrm{rad}(\Delta(\Lambda^i\mathfrak{h}))$. □

Corollary: If $c=\frac{1}{h}+m$, then $\sum_{i=0}^{n}(-1)^i\,\mathrm{Tr}_{e_\epsilon\Delta(\Lambda^i\mathfrak{h})}(t^h)=1$.

In type A, we proved this using a Koszul complex.

$x\partial_\epsilon \to x\partial_e,\ y e_\epsilon \mapsto y e$
$h e_\epsilon \mapsto h e$

Proposition: $\forall c \in \mathbb{C}[R]^W$, $e_\epsilon H_c(W) e_\epsilon \cong e H_{c-1}(W) e$

This is related to the theory of shift functors. The proof uses the deformed Harish-Chandra homomorphism.

Theorem: Set $c = \frac{1}{h} + 1$. $L(\text{triv})$ is finite dimensional and its Hilbert series is $\mathrm{Tr}_{L(\text{triv})}(t^h) = t^{-|R_+|}(1+t+t^2+\dots+t^h)^n$.

Proof: $eH_{\frac{1}{h}}(W)e$ has a one-dimensional module $e\mathbb{C}$, hence so does $e_\epsilon H_c(W) e_\epsilon$ by the proposition above.

$L = H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} \mathbb{C}$ is a finite dimensional $H_c(W)$-module because $H_c(W)$ is a finite module over $e_\epsilon H_c e_\epsilon$

It is also simple because $e_\epsilon L \cong \mathbb{C}$ and $e_\epsilon L(\tau) \neq 0$ $\forall \tau \in \mathrm{Irr}(W)$ as proved above. (If L had at least two simple composition factors, $e_\epsilon L$ would have dimension ≥ 2.)

L has a copy of $\Lambda^n \mathfrak{h}$ in degree 0 on which $\underline{h}_\epsilon$ acts by multiplication by 0 since $\underline{h}e \mapsto h e_\epsilon$ under $eH_{c+1}(W)e \xrightarrow{\sim} e_\epsilon H_c(W) e_\epsilon$

For which $\Delta(\Lambda^i \mathfrak{h})$ does $\Lambda^n \mathfrak{h}$ appear in degree 0?

Recall
$$\mathrm{Tr}_{e_\epsilon \Delta(\Lambda^i \mathfrak{h})}(t^h) = t^{-|R_+| + i(h+1)}\, p(t) \sum_{1\le j_1 < \dots < j_{n-i} \le n} t^{e_{j_1} + \dots + e_{j_{n-i}}}$$

(since $\sum_{i=1}^n e_i = |R_+|$)

$$= t^{i(h+1)}\, p(t) \sum_{1\le j_1 < \dots < j_i \le n} t^{-e_{j_1} - \dots - e_{j_i}}$$

$$= t^i\, p(t) \sum_{1\le j_1 < \dots < j_i \le n} t^{(h-e_{j_1}) + \dots + (h - e_{j_i})}$$

$e_j < h$ $\forall 1 \le j \le n$, so t^0 appears in $\mathrm{Tr}_{e_\epsilon \Delta(\Lambda^i \mathfrak{h})}(t^h) \iff i = 0$.

$\Longrightarrow L \cong L(\mathrm{Triv})$.

$e_\epsilon L \cong \mathbb{C}$, so $[e_\epsilon L] = \sum_{i=0}^{n} (-1)^i [e_\epsilon \Delta(\Lambda^i \mathfrak{h})]$ in $K(\text{mod-}e_\epsilon H_c e_\epsilon)$

since the two sides have the same Hilbert series (as proved in a previous lemma). Note that the Hilbert series of the $e_\epsilon \Delta(\Lambda^i \mathfrak{h})$ have distinct lowest degrees.

$$\Longrightarrow [L(\mathrm{triv})] = \sum_{i=0}^{n} (-1)^i [\Delta(\Lambda^i \mathfrak{h})] \text{ in } K(\text{mod-}H_c(W))$$

From this, one can calculate $\mathrm{Tr}_{L(\mathrm{Triv})}(t^h)$. $\square$

Corollary: $\dim_{\mathbb{C}} L(\mathrm{triv}) = (h+1)^n$.

Application to diagonal harmonics:

Ring of diagonal coinvariants: $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] / (\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+)$.

Theorem: There exists a W-stable quotient ring R_W of D_W such that
(i) $\dim R_W = (h+1)^n$
(ii) The image of $\mathbb{C}[\mathfrak{h}]$ in R_W is isomorphic to $\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv}\, W}$

Proof: Let $L = H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} \mathbb{C}$ for $c = \frac{1}{h} + 1$ and set $R_W = \mathrm{gr}\, L \otimes \Lambda^n \mathfrak{h}$.
R_W has the same Hilbert series as L, so $\dim R_W = \dim L = (h+1)^n$.