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Fundamental results about $H_{t=0,c}(W)$

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"$H_c(W)$"

Let Z_c be the center of $H_{t=0,c}(W)$. The center of $H_{t,c}(W)$ is trivial when $t\neq 0$ whereas $\mathrm{Spec}(Z_c)$ is a complicated algebraic variety.

In type A, it is the Calogero-Moser space:

When $W=S_n$, $\underbrace{}_{\text{and } c\neq 0}$ $\mathrm{Spec}(Z_c)\cong \{(X,Y)\in \mathfrak{gl}_n\times\mathfrak{gl}_n \mid \mathrm{rank}([X,Y]+Id)=1\}/\!/PGL_n$

This is one particular example of a quiver variety.

For M a simple $H_c(W)$-module, Schur's lemma says that $z\in Z_c$ acts on M by a scalar, so to M we can associate a character $\chi: Z_c\to\mathbb{C}$.

Given such a χ, we would like to construct a simple H_c-module. $\updownarrow$ (closed) point on $\mathrm{Spec}\,Z_c$

Satake isomorphism: The map $Z_c\to eH_ce$, $z\mapsto ze$, is an algebra isomorphism. (In particular, eH_ce is commutative!)

Classical invariant theory $\Rightarrow$ $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ is a finitely generated module over $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$. Since $gr(H_ce)\cong\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ and $gr(eH_ce)\cong\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$ (PBW property), H_ce is a finitely generated (right) eH_ce-module.

If $\chi: Z_c\to\mathbb{C}$, we can view χ as a one-dimensional eH_ce-module via the Satake isomorphism. $H_ce\otimes_{eH_ce}\chi$ is a left H_c-module and it is finite dimensional

even simple if $\mathrm{Spec}(Z_c)$ is **smooth** because,
in this case, the following functors are equivalences of categories:

$$H_c\text{-mod} \underset{F}{\overset{G}{\rightleftarrows}} eH_ce\text{-mod} \quad F(M) = H_ce \otimes_{eH_ce} M,\ G(N) = eN$$

These two functors are equivalences $\Longleftrightarrow$ $H_ceH_c = H_c$, which can be hard to check.

$\mathrm{Spec}(Z_c)$ is not always smooth, but it is when $W = S_n$. When it is not smooth, it is still possible to do the previous construction by restriction to the smooth locus.

Moreover, if $\mathrm{Spec}(Z_c)$ is smooth, then any simple H_c-module is isomorphic to $\mathbb{C}[W]$ as a W-module (regular representation).

These results about simple modules were obtained by showing first that
I the eH_ce-module H_ce defines a coherent sheaf on $\mathrm{Spec}(Z_c)$ which is a vector bundle over the smooth locus.

Note that there is always an obvious map $H_{t,c} \longrightarrow \mathrm{End}_{eH_{t,c}e}(H_{t,c}e)$.

II Proposition: $\forall t,c$, this map is an algebra isomorphism.

III Proposition: If V is a vector bundle on an affine variety $X = \mathrm{Spec}(A)$, then $\mathrm{End}_{\mathcal{O}_X}(V)$ is Morita equivalent equivalent to A.

When $\mathrm{Spec}(Z_c)$ is smooth, **I, II, III** $\Rightarrow$ H_c is Morita equivalent to Z_c ($\simeq eH_ce$) $\Longrightarrow$ The simple H_c-modules are classified by the closed points of $\mathrm{Spec}(Z_c)$.

Basic representation theory of $H_{t=0,c}(W)$

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Contrary to $H_{t=1,c}(W)$, $H_{t=0,c}(W)$ has infinitely many irreducible finite dimensional representations. " $H_c(W)$

W: complex reflection group $\mathfrak{h}$: reflection representation.
S: set of complex reflections of W. ω: canonical symplectic form on $\mathfrak{h}\oplus\mathfrak{h}^*$.
ω_s: skew-symmetric form on $\mathfrak{h}\oplus\mathfrak{h}^*$ which coincides with ω on $\mathrm{Im}(\mathrm{Id}-s)$ and has $\mathrm{Ker}(\mathrm{Id}-s)$ as its radical. $c: S/_W \to \mathbb{C}$.

In $H_c(W)$: $x\in\mathfrak{h}^*, y\in\mathfrak{h}$, $\quad xy-yx = \sum_{s\in S} c_s\omega_s(y,x)s$

Let $A = \mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]^W \subset H_c(W)$

Lemma: $A \subset Z_c$ = center of $H_c(W)$. Advantage of A ($\mathrm{Spec}\,A$): it is smooth and much easier to describe than Z_c.

Proof: Set $\mathfrak{h}_s = \{y\in\mathfrak{h} \mid s(y)=y\}$. Choose $\alpha_s^\vee$ such that $\mathfrak{h} = \mathfrak{h}_s \oplus \mathbb{C}\alpha_s^\vee$ and $\alpha_s^\vee$ is an eigenvector for s.

$$\omega_s(y,x) = \frac{\alpha_s(y)\,\alpha_s^\vee(x)}{\alpha_s(\alpha_s^\vee)}.$$

Set $\lambda_s = 1 - e^{2\pi i n_s} \in \mathbb{C}$, $s(\alpha_s^\vee) = e^{2\pi i n_s}\alpha_s^\vee$, $n_s\in\mathbb{Q}$. $y\in\mathfrak{h}$, $y = y_s + y_s^\perp$, $y_s\in\mathfrak{h}_s$, $y_s^\perp \in \mathbb{C}\alpha_s^\vee$

$$s(y) = y_s + s(y_s^\perp) = y_s + e^{2\pi i n_s} y_s^\perp = y - \lambda_s y_s^\perp = y - \lambda_s \frac{\alpha_s(y)}{\alpha_s(\alpha_s^\vee)}\alpha_s^\vee$$

More generally, $s(p) = p - \lambda_s \frac{\alpha_s(p)}{\alpha_s(\alpha_s^\vee)} \alpha_s^\vee \quad \forall p \in \mathbb{C}[\mathfrak{h}^*]$ if we

define α_s inductively by $\alpha_s(p_1p_2) = \alpha_s(p_1)p_2 + p_1\alpha_s(p_2) - \lambda_s \frac{\alpha_s(p_1)\alpha_s(p_2)}{\alpha_s(\alpha_s^\vee)}\alpha_s^\vee$.

We can now check by induction that, if $p = p_1p_2 \in \mathbb{C}[\mathfrak{h}^*]$,

$$\begin{aligned}
[x,p] &= [x,p_1]p_2 + p_1[x,p_2] \\
&= \sum_{s\in S} c_s \frac{\langle x,\alpha_s^\vee\rangle}{\alpha_s(\alpha_s^\vee)} \alpha_s(p_1) s\, p_2 + p_1 \sum_{s\in S} c_s \frac{\langle x,\alpha_s^\vee\rangle}{\alpha_s(\alpha_s^\vee)} \alpha_s(p_2) s \\
&= \sum_{s\in S} c_s \frac{\langle x,\alpha_s^\vee\rangle}{\alpha_s(\alpha_s^\vee)} \left(\alpha_s(p_1)\left(p_2 - \lambda_s \frac{\alpha_s(p_2)}{\alpha_s(\alpha_s^\vee)}\alpha_s^\vee\right)s + p_1\alpha_s(p_2)s\right) \\
&= \sum_{s\in S} c_s \frac{\langle x,\alpha_s^\vee\rangle}{\alpha_s(\alpha_s^\vee)} \alpha_s(p)\, s.
\end{aligned}$$

If $p \in \mathbb{C}[\mathfrak{h}^*]^W$, then $p = s(p)\ \forall s\in S \Rightarrow \alpha_s(p) = 0\ \forall s \in S \Rightarrow [x,p] = 0$.
$\Rightarrow p \in Z_c$. The same argument applies to $\mathbb{C}[\mathfrak{h}]^W$. $\square$ $\quad \forall x \in \mathfrak{h}^*$

$$A \subset Z_c \Rightarrow \Upsilon : \mathrm{Spec}(Z_c) \longrightarrow \mathrm{Spec}(A) = \mathfrak{h}/W \times \mathfrak{h}^*/W$$

This is a dominant morphism.

The geometry of Υ is related to the representation theory of $H_c(W)$.

(reduced) scheme theoretic fiber: $\Upsilon^*(q)$, $q \longleftrightarrow$ maximal ideal of A. $0 \longleftrightarrow \mathbb{C}[\mathfrak{h}]^W_+ \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W_+$.

Corollary: $H_c(W)$ is a finite module over its center (because it is a finite module over A: $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}} \mathbb{C}[W] / (\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W)_+ \cong \underbrace{\mathbb{C}[\mathfrak{h}]^{\mathrm{coinv} W} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]^{\mathrm{coinv} W} \otimes_{\mathbb{C}} \mathbb{C}[W]}_{\dim_{\mathbb{C}} = |W|^3}$)

We will be interested in $\Upsilon^*(0)$.

When $W = S_n$, $c \neq 0$, $\mathrm{Spec}(Z_c)$ is the Calogero-Moser space and, for $(X,Y) \in \mathrm{Spec}(Z_c)$, $\Upsilon(X,Y)$ are the eigenvalues of $X, Y \in \mathfrak{gl}_n$.

Baby Verma modules

Set $\overline{H}_c(W) = \dfrac{H_c(W)}{(A_+)}$. This is a finite dimensional algebra, $\overline{H}_c(W) \cong \mathbb{C}[\mathfrak{h}]^{coinvW} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]^{coinvW} \otimes_{\mathbb{C}} \mathbb{C}[W]$ as vector spaces.

$\mathbb{C}[\mathfrak{h}^*]^{coinvW} \rtimes W \hookrightarrow \overline{H}_c(W)$. For $\rho \in \mathrm{Irr}\, W$, set

$M(\rho) = \overline{H}_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coinvW} \rtimes W} \rho$, ρ being a $\mathbb{C}[\mathfrak{h}^*]^{coinvW} \rtimes W$-module by pulling back via $\mathbb{C}[\mathfrak{h}^*]^{coinvW} \rtimes W \to \mathbb{C}[W]$. $M(\rho) \cong \mathbb{C}[\mathfrak{h}]^{coinvW} \otimes_{\mathbb{C}} \rho$

$p \otimes w \longmapsto p(0) w$ as vector spaces

$\deg(y) = 1$, $\deg(x) = -1$. $\overline{H}_c(W)$ is graded and so is $M(\rho)$ (ρ has degree 0)

Fake degrees: $f_\rho(t) = \sum_{i \in \mathbb{Z}} \left(\mathbb{C}[\mathfrak{h}^*]^{coinvW} : \rho[i] \right) t^i$

↳ multiplicity of ρ in degree i.

In $K(\text{gr-mod}\,\mathbb{C}[W])$, $[M(\rho)] = \sum_{\tau \in \mathrm{Irr} W} f_\tau(t) [\rho \otimes \tau]$

gr-mod-$\overline{H}_c(W)$ is not a highest weight category (there is no partial ordering with the required properties), but for each $\rho \in \mathrm{Irr}\, W$ we have $M(\rho)$, a simple module $L(\rho)$ and its projective cover $P(\rho)$ and the following properties hold.

Proposition: 1) $M(\rho) / \mathrm{rad}(M(\rho)) \cong L(\rho)$

2) $\{ L(\rho)[i] \mid \rho \in \mathrm{Irr}\, W,\ i \in \mathbb{Z} \}$ is a complete set of pairwise non-isomorphic simple graded $\overline{H}_c(W)$-modules.

Set $M'(\rho) = \left(\rho^\vee \otimes_{\mathbb{C}[\mathfrak{h}]^{\text{co}W} \rtimes W} \overline{H}_c(W) \right)^\vee$ ↳ graded vector space dual

Brauer-type reciprocity formula:

$$[P(\rho) : M(\tau)[i]] = [M'(\tau)[i] : L(\rho)]$$

Implicitly: this multiplicity is well-defined.

If W is a Coxeter group, $[M'(\tau)[i] : L(\rho)] = [M(\tau) : L(\rho)]$.
($\rho \cong \rho^\vee$ in this case)

Scheme theoretic fiber $\Upsilon^*(0) = \mathrm{Spec}\left(Z_c / A_+ Z_c \right)$ (A_+ is the maximal ideal of A corresponding to 0)

The closed points of $\Upsilon^*(0)$ are the maximal ideals M of Z_c such that $A_+ Z \hookrightarrow M$, $M \cap A = A_+$. There are finitely many such ideals.

Set $\mathcal{O}_M = (Z_c)_M / A_+ (Z_c)_M$ (localization)

Some general results about triples like $A \subset Z_c \subset H_c(W)$

Def: The Azumaya locus of $H_c(W)$ is the set of maximal ideals M of Z_c for which $H_c(W) / M H_c(W)$ is a simple Artinian ring.
(In particular, it has a unique simple module.)

Let $m \subset A$ be a maximal ideal, $m \subset M$: maximal ideal of Z_c. The index of ramification of M with respect to m is the least positive integer i such that $M^i \subset m Z_c$. M is called unramified if $M = (M \cap A) Z_c$. The unramified locus of Z_c with respect to A is the set of all such ideals.

[BG] Lemma: The unramified locus of $\mathrm{Spec}(Z_c)$ is contained in the smooth locus.

The Azuma locus of $H_c(W)$ can also be described as the set of closed points $P \in \mathrm{Spec}(Z_c)$ for which there exist simple modules L on which $I(P)Z$ acts by 0 (so the Z_c-module structure on L factors through $Z_c / I(P) Z_c$) and which are of maximal dimension. It is an open subset of $\mathrm{Spec}(Z_c)$. The maximal dimension is $|W|$.

Theorem (Etingof–Ginzburg): The Azumaya points are precisely the smooth points of $\mathrm{Spec}(Z_c)$.

Let m be a maximal ideal of A.

Proposition [BG]: The blocks of $\overline{H}_c(W) / m \overline{H}_c(W)$ are in bijection with the maximal ideals of Z_c which contain m.

Reminder: $\overline{H}_c(W)$ is a direct sum of indecomposable algebras which are called blocks. The blocks are in bijection with the primitive central idempotents of $\overline{H}_c(W)$.

If $m = 0$, we get $\overline{H}_c(W) = \bigoplus_{M \in \Upsilon^{-1}(0)} B_M$. If M is an Azumaya point of $\mathrm{Spec}(Z_c)$, then $B_M \cong M_k(\mathcal{O}_M)$ $k = |W|$.

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The baby Verma module $M(\rho)$ is indecomposable, hence belongs to a unique block of $\overline{H}_c(W)$. This gives a map $\Theta: \mathrm{Irr}\, W \longrightarrow \Upsilon^{-1}(0)$.

If $M \in \Upsilon^{-1}(0)$ is a smooth point of $\mathrm{Spec}(Z_c)$, then $\Theta^{-1}(M)$ is unique and $B_M \cong \mathrm{Mat}_{|W|}(\mathbb{C}_W)$.

$\Rightarrow$ If $\mathrm{Spec}(Z_c)$ is smooth (e.g. if $W = S_n$), then Θ is a bijection.