Fundamental results about $H_{t=0,c}(W)$

P. Etingof, V. Ginzburg: Symplectic reflection algebras, $H_{t,c}(W)$

Let $Z_c$ be the center of $H_{t=0,c}(W)$. The center of $H_{t,c}(W)$ is trivial when $t \neq 0$, whereas $\text{Spec}(Z_c)$ is a complicated algebraic variety.

In type $A$, it is the Calogero–Moser space:

$$\text{Spec}(Z_c) = \{(X,Y) \in \mathfrak{g}_n \times \mathfrak{g}_n | \text{rank}(\mathbf{X}Y + \text{Id}) = 1\}$$

This is one particular example of a quiver variety.

For $M$ a simple $H_{c}(W)$-module, Schur's lemma says that $Z_c$ acts on $M$ by a scalar, so to $M$ we can associate a character $\chi : Z_c \to \mathbb{C}$.

Given such a $\chi$, we would like to construct a simple $H_{c}$-module.

Satake isomorphism: The map $Z_c \to \text{et}_{H_{c}}$ is an algebra isomorphism. (In particular $\text{et}_{H_{c}}$ is commutative!)

Classical invariant theory $\to$ $[\mathbb{C}[\mathfrak{h}]^*]$ is a finitely generated module over $[\mathbb{C}[\mathfrak{h}]^*]^W$. Since $\text{gr}(H_{c}) \cong [\mathbb{C}[\mathfrak{h}]^*]$ and $\text{gr}(H_{c}) \subseteq [\mathbb{C}[\mathfrak{h}]^*]^W$ (PBW property), $H_{c}$ is a finitely generated (right) $\text{et}_{H_{c}}$-module.

If $X : Z_c \to \mathbb{C}$, we can view $X \mathbf{c}$ as a one-dimensional $\text{et}_{H_{c}}$-module via the Satake isomorphism. $H_{c} \otimes \text{et}_{H_{c}} X$ is a left $H_{c}$-module and it is finite dimensional.
even simple if $\text{Spec}(\mathbb{Z}_e)$ is smooth because, in this case, the following functors are equivalences of categories:

$$H^*_c \rightarrow A^*_c \rightarrow M \rightarrow \mathbb{E} \rightarrow \mathbb{E}^*_c \rightarrow H^*_c \rightarrow H^*_c$$

These two functors are equivalences $\iff H^*_c = H^*_c$ which can be hard to check.

$\text{Spec}(\mathbb{Z}_e)$ is not always smooth, but it is when $W = S_n$. When it is not smooth, it is still possible to do the previous construction by restriction to the smooth locus.

Moreover, if $\text{Spec}(\mathbb{Z}_e)$ is smooth, then any simple $H^*_c$-module is isomorphic to $\mathbb{Z}_W$ as a $W$-module (regular representation).

These results about simple modules were obtained by showing first that the $\mathbb{E}^*_c$-module $H^*_c \mathbb{E}^*_c$ defines a coherent sheaf on $\text{Spec}(\mathbb{Z}_e)$ which is a vector bundle over the smooth locus.

Note that there is always an obvious map $H^*_c \rightarrow \mathbb{E}^*_c H^*_c \mathbb{E}^*_c$.

**II Proposition:** $H^*_c$, this map is an algebra isomorphism.

**III Proposition:** If $V$ is a vector bundle on an affine variety $X = \text{Spec}(A)$, then $\mathbb{E}^*_c(V)$ is $H^*_c$-torsion equivalent equivalent to $A$.

When $\text{Spec}(\mathbb{Z}_e)$ is smooth, $\text{I, II, III} \Rightarrow H^*_c$ is $H^*_c$-torsion equivalent to $\mathbb{Z}_e \otimes H^*_c$, the simple $H^*_c$-modules are classified by the closed points of $\text{Spec}(\mathbb{Z})$. 
Basic representation theory
of $M_{t=0,c}(W)$

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Contrary to $H_{t=0,c}(W)$, $H_{t=0,c}(W)$ has infinitely many irreducible
finite dimensional representations. \hspace{1cm} H_c(W)

$W$: complex reflection group \hspace{1cm} b_i$: reflection representation.

$S$: set of complex reflections of $W$. \hspace{1cm} \omega$: canonical symplectic form on $b_{x,y}^*$.

$\omega_S$: skew-symmetric form on $b_{x,y}^*$ which coincides with $\omega$ on $\text{Im}(\text{Id}-S)$
and has ker $(\text{Id}-S)$ as its radical. \hspace{1cm} C: \frac{S}{W} \rightarrow C.

$\mathfrak{H}_c(W)$: $x_{\nu}, y_{\nu}, x_{\nu}y_{\nu} = \sum_{s \in S} \epsilon_s \omega_s(x, y) s$

Let $A = \{ [b] \}^W_0 \text{C} \{ [b] \}^W_c \hspace{1cm} c \mathcal{H}_c(W)$

Lema: $A \subset Z_c = \text{center of } H_c(W)$.\hspace{1cm} Advantage of $A$: this smooth and

much easier to describe than $Z_c$.

Proof: Set $b_s = \{ y \in b | s(y) = y \}$. Choose $\alpha_s^y$ such that $b_s = b_s \circ \text{C} \alpha_s^y$

then $\alpha_s$ is an eigenvector for $s$.

$\omega_s(x, y) = \frac{\alpha_s(y) - \alpha_s^y(x)}{s}$

Set $\lambda_s = \alpha_y^y - \frac{\alpha_s(\alpha_s^y)}{s}$, $\alpha_s(\alpha_s^y) = \epsilon_{\min_y} \alpha_s^y$, $s \in \{ y \in b \}$, $y_{\nu} \neq y_{\nu}$, $\epsilon_{\min_y} \alpha_s^y$, $\nu \in \mathbb{Q}$. \hspace{1cm} y_{\nu} = y_{\nu}, y_{\nu} \neq y_{\nu}$, $s(y) = y_{\nu}, y_{\nu}^s = y_{\nu}$

Let $\lambda_s = \alpha_y^y - \frac{\alpha_s(\alpha_s^y)}{s}$, $\alpha_s(\alpha_s^y) = \epsilon_{\min_y} \alpha_s^y$, $s \in \{ y \in b \}$, $y_{\nu} \neq y_{\nu}$, $\epsilon_{\min_y} \alpha_s^y$, $\nu \in \mathbb{Q}$.
More generally, \( s(p) = p - \lambda s(\alpha_s(p)) \alpha_s \) for any \( p \in \mathcal{L} \).

Define \( \alpha_s \) inductively by \( \alpha_s(p, p_1) = \alpha_s(p) p_1 + p_1 \alpha_s(p_1) - \lambda s(\alpha_s(p)) \alpha_s(\alpha_s(p_1)) \).

We can now check by induction that, if \( p = p_1 p_2 \in \mathcal{L} \),

\[
[x, p] = [x, p_1] p_2 + p_1 [x, p_2]
\]

\[
= \sum_{s \in S} \frac{c_s \langle x, \alpha_s \rangle}{\alpha_s(\alpha_s)} \alpha_s(p_1) + p_1 \sum_{s \in S} \frac{c_s \langle x, \alpha_s \rangle}{\alpha_s(\alpha_s)} \alpha_s(p_2)
\]

\[
= \sum_{s \in S} \frac{c_s \langle x, \alpha_s \rangle}{\alpha_s(\alpha_s)} \alpha_s(p_1) + p_1 \sum_{s \in S} \frac{c_s \langle x, \alpha_s \rangle}{\alpha_s(\alpha_s)} \alpha_s(p_2)
\]

If \( p \in \mathcal{L}^\times \), then \( x(s(p), ) \forall s \in S \) \( \Rightarrow \alpha_s(p) = 0 \) \( \forall s \in S \) \( \Rightarrow [x, p] = 0 \).

\( \Rightarrow p \in \mathcal{L} \). The same argument applies to \( \mathcal{L}^\times \).

\( \forall x \in \mathcal{L}^\times \)

\( A \subset \mathcal{L} \Rightarrow \mathcal{L} : \text{Spec}(\mathcal{L}) \to \text{Spec}(A) = \mathfrak{b}/\mathfrak{w} \times \mathfrak{b}/\mathfrak{w} \)

This is a dominant morphism.

The geometry of \( \mathcal{L} \) is related to the representation theory of \( H_c(\mathcal{L}) \). (product scheme theoretic fibers: \( \mathcal{L}^\times \), \( q \mapsto \text{maximal ideal of } A \), \( q \mapsto \mathcal{C}[\mathcal{L}]^\times \).

Corollary: \( H_c(\mathcal{L}) \) is a finite module over its center (because it is a finite module over \( A \): \( \mathcal{C}[\mathcal{L}]^\times \).

We will be interested in \( \mathcal{L}^\times(0) \).

When \( W = S_n \), \( \text{Spec}(\mathcal{L}) \) is the Categalo-Howe space and, for \( (X, Y) \in \text{Spec}(\mathcal{L}) \),

\( X(Y, X) \) are the eigenvalues of \( X, Y \) each.
Baby Verma modules

Set \( H_c(W) = \frac{H_c(W)}{(A_1)} \). This is a finite dimensional algebra, \( H_c(W) = \mathbb{C}[\mathfrak{b}] \otimes \mathbb{C}[\mathfrak{b}] \otimes \mathbb{C}[W] \) as vector spaces.

\( \mathbb{C}[\mathfrak{b}] \otimes W \rightarrow H_c(W) \). For \( p \in \text{Irr} W \), set

\[
M(p) = H_c(W) \otimes_{\mathbb{C}[\mathfrak{b}] \otimes W} p, \quad p \text{ being a } \mathbb{C}[\mathfrak{b}] \otimes W \text{-module}
\]

by pulling back via \( \mathbb{C}[\mathfrak{b}] \otimes W \rightarrow \mathbb{C}[W] \). \( M(p) \cong \mathbb{C}[\mathfrak{b}] \otimes W \)

\[
p \otimes w \mapsto p(w) w
\]
as vector spaces.

\( \deg(y) = 1, \deg(x) = -1 \). \( H_c(W) \) is graded and so is \( M(p) \) (\( p \) has degree).

Fake degree:

\[
f_p(t) = \sum_{i \in \mathbb{Z}} (\mathbb{C}[\mathfrak{b}] \otimes W : p_i) t^i
\]

is multiplicity of \( p \) in degree \( i \).

In \( K(\text{gr-mod} \mathbb{C}[W]) \),

\[
[M(p)] = \sum_{r \in \text{Irr} W} f_r(t) [p \otimes r]
\]

\( \text{gr-mod} \mathbb{C}[W] \) is not a highest weight category (there is no partial ordering with the required properties), but for each \( p \in \text{Irr} W \) we have \( M(p) \), a simple module \( L(p) \) and its projective cover \( P(p) \) and the following properties hold.
Proposition: 1) \[ M(\rho)_{\text{red}}(W) \cong L(\rho) \]

2) \[ \{ L(\rho)_{\text{red}}(W), \rho \in \text{Irr}(W), \rho \in \mathbb{Z} \} \]

is a complete set of pairwise non-isomorphic simple graded \( H_c(W) \)-modules.

Set \( M'(\rho) = \left( \rho \otimes_{\mathcal{Z}(W)} H_c(W) \right)^{\vee} \) is graded vector space dual.

Brauer-type reciprocity formula:

\[ [P(\rho); M(\gamma)_{\text{red}}] = [M'(\rho)^{\vee}; L(\rho)] \]

Implicity, this multiplicity is well-defined.

If \( W \) is a Coxeter group, \[ [M'(\rho)^{\vee}; L(\rho)] = [M(\rho); L(\rho)], \]

(\( \rho = \rho' \) in this case)

Scheme theoretic fiber \( \gamma^*(0) = \text{Spec} \left( \mathbb{Z}/A+\mathbb{Z} \right) \) (\( A \) is the maximal ideal of \( A \) corresponding to \( A \))

The closed points of \( \gamma^*(0) \) are the maximal ideals \( M \) of \( \mathbb{Z} \) such that \( A+\mathbb{Z} \subset M \), \( M \cap A = A+\mathbb{Z} \). There are finitely many such ideals.

Set \( O_M = (\mathbb{Z}/A+\mathbb{Z})_M \) (localization)

Some general results about triples like \( A+\mathbb{Z} \in H_c(W) \)

Def: The Azumaya locus of \( H_c(W) \) is the set of maximal ideals \( M \) of \( \mathbb{Z} \) for which \( H_c(W)/M \) is a simple Artinian ring.

(In particular, it has a unique simple module)
Let \( m \in A \) be a maximal ideal, \( m \subset M : \) maximal ideal of \( \mathbb{Z}_c. \) The index of ramification of \( M \) with respect to \( m \in A \) is the least positive integer \( i \) such that \( M^i : m \cdot \mathbb{Z}_c. \) \( M \) is called unramified if \( M = (M : m \mathbb{Z}_c). \) The unramified locus of \( \mathbb{Z}_c \) with respect to \( A \) is the set of all such ideals.

Lemma: The unramified locus of \( \text{Spec}(\mathbb{Z}_c) \) is contained in the smooth locus.

The Azumaya locus of \( H_c(W) \) can also be described as the set of closed points \( P \in \text{Spec}(\mathbb{Z}_c) \) for which there exist simple modules \( L \) on which \( \mathcal{I}(P) \mathbb{Z}_c \) acts by \( 0 \) (so the \( \mathbb{Z}_c \)-module structure on \( L \) factors through \( \mathbb{Z}_c / \mathcal{I}(P) \mathbb{Z}_c \) ) and which are of maximal dimension. It is an open subset of \( \text{Spec}(\mathbb{Z}_c) \). The maximal dimension is \( |W|. \)

Theorem (Etingof-Ginzburg): The Azumaya points are precisely the smooth points of \( \text{Spec}(\mathbb{Z}_c). \)

Let \( m \) be a maximal ideal of \( A. \)

Proposition: The blocks of \( H_c(W) / mH_c(W) \) are in bijection with the maximal ideals of \( \mathbb{Z}_c \) which contain \( m. \)

Reminder: \( H_c(W) \) is a direct sum of indecomposable algebras which are called blocks. The blocks are in bijection with the primitive central idempotents of \( H_c(W). \)

If \( m = 0, \) we get \( H_c(W) = \bigoplus_{M \in \text{spec}(W)} B_M. \) If \( M \) is an Azumaya point of \( \text{Spec}(\mathbb{Z}_c), \) then \( B_M = M_k(\Omega_M) \) for \( k = |W|. \)
The baby Verma module $M(p)$ is indecomposable, hence belongs to a unique block of $\mathcal{H}_c(W)$. This gives a map $\Theta: \text{Irr } W \to \gamma^{-1}(0)$.

If $M \in \gamma^{-1}(0)$ is a smooth point of $\text{Spec}(Z_c)$, then $\Theta^{-1}(M)$ is unique.

$\Rightarrow$ If $\text{Spec}(Z_c)$ is smooth (e.g. if $W = S_n$), then $\Theta$ is a bijection.

and $B_M = \text{Net}_{\text{W}}(M)$. 