Literature

I Dunkl operators for complex reflection groups

(joint with C. Dunkl; Proc. LMS (3) 86 (2003))

II Category $\mathcal{O}$ for rational Cherednik algebras

(joint with Ginzburg, Guay & Rouquier; Inv. Math ... (2003))

I gives a natural construction of the rational Cherednik algebra.
II gives the natural context for the main problem that arises as a result of the construction in I

III Quasi-invariants of complex reflection groups

(unpublished preprint (to appear) of Yuri Berest & Oleg Chalykh)

PLANNING

1) Dunkl representation of the Cherednik algebra
(Dunkl operators, Dunkl pairing, Singular parameters, shift operators, Spherical algebra)
2) KZ-functor (Localization, monodromy Hecke algebra, duality, main results KZ-functor, decomposition numbers)

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3) Quasi-invariants (differential operators on quasi-invariants, symmetries spherical algebras, existence shift operators
4) KZ-twist & category O (Fake degrees)

I. Complex reflection groups

K field of char. 0

V fin. diml. vector space / K

DEF $g \in GL(V)$ is called a pseudo reflection if g has finite order, and if $\mathrm{Ker}(g-\mathbb{1})$ is a hyperplane in V

DEF $G \subset GL(V)$ finite group. G is called a pseudo reflection group if G is generated by pseudo reflections.

THM (Chevalley; Shephard-Todd) $G \subset GL(V)$ finite. The following are equivalent:

(i) $S(V)^G$ is a polynomial algebra.

(ii) G is a pseudo-reflection group.

(iii) $S(V)$ is a free, rank one $S(V)^G[G]$ module

- If $K = \mathbb{Q}$: G is a (finite) Weyl group
- If $K \subseteq \mathbb{R}$: G is a real reflection group
- If $K \subseteq \mathbb{C}$: G is a complex refl. gp.

THM (Bessis '98) K is splitting field for G

Classification (Shephard-Todd)

$G \subset GL(V)$ irreducible CRG. G isom. to:

(I) $\mathfrak{S}_{n+1}$ acting on $V = \{ x \in \mathbb{C}^{n+1} \mid \sum x_i = 0 \}$.

(II) $G(m,p,n) \subset GL_n(\mathbb{C})$ with $m, p, n \in \mathbb{N}$, $m > 1$, $p \geq 1$, $n \geq 1$ and $p \mid m$.

$$G(m,p,n) \subset \underbrace{(\mathbb{Z}/m\mathbb{Z})^n}_{\substack{\text{diagonal with} \\ \text{entries } m\text{-th roots of } 1}} \rtimes \underbrace{\mathfrak{S}_n}_{\substack{\text{permutation} \\ \text{matrices}}} = G(m,1,n)$$

$$G(m,p,n) = \{ g = D \times \sigma \mid (\det(D))^{m/p} = 1 \}$$

(III) One of 34 exceptional cases; the largest is E_8; many rank 2 cases related to the finite subgroups $\Gamma \subset SU(2)$ by central ext.: $G = \left\{ \begin{pmatrix} \alpha^{+1} & 0 \\ 0 & \alpha^{+1} \end{pmatrix} \mid \alpha^n = 1 \right\} \cdot \Gamma$ (n suitable)

Remark (Shephard-Todd) G irreducible CRG of rank n. Then G can be generated by a set S of n or $n+1$ pseudoreflections with relations of the form (I) homogeneous relations (braid relations) and (II) finite order rels ($s_i^{e_i} = 1$)

Ex. $G(m,p,n)$ needs $|S| = n+1 \Leftrightarrow n > 1$, $p \neq 1, m$.

Braid groups

From now on $K = \mathbb{C}$

$G \subset GL(V)$ a CRG. Let $\mathcal{A}$ denote the arrangement of reflection hyperplanes of G. Put

$$\boxed{V^{reg} = V \setminus \bigcup_{H \in \mathcal{A}} H}$$

G acts freely on V^{reg} (Steinberg)

Put $\boxed{X^{reg} = G \backslash V^{reg} \ (\simeq \mathbb{C}^n \setminus \{\Delta_G = 0\})}$

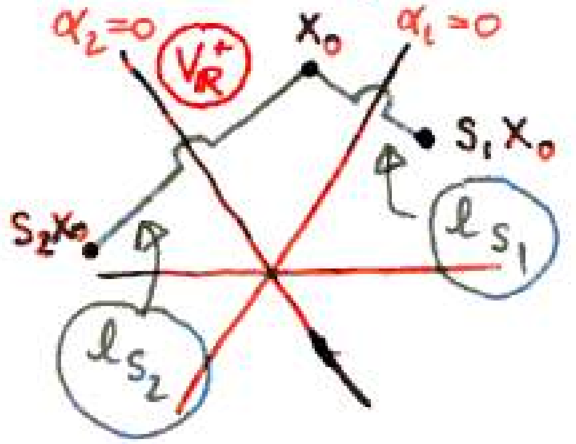
for the regular orbit space of G

DEF Choose $x_0 \in X^{reg}$. The group $B^G = \pi_1(X^{reg}, x_0)$ is called "the" topological braid group of G.

If $W \subset GL(V_{\mathbb{R}})$ is a real reflection group then W admits a Coxeter system (W, S)

DEF The braid group $B^{(W,S)}$ of (W,S) is the group with generators σ_s $(s \in S)$ and relations given by the braid relations only

<u>THM</u> (Brieskorn; Deligne) Choose closed loops ℓ_s ($s \in S$) in X^{reg} as follows:



Then the ~~map~~ assignment

$$B^{(W,S)} \ni \sigma_s \to \ell_s \in B^W$$

extends to an isomorphism

$$B^{(W,S)} \xrightarrow{\sim} B^W$$

<u>DEF</u> G a CRG. A generator system (G,S) is called "Coxeter-like" if there exists a similar isomorphism $B^{(G,S)} \xrightarrow{\sim} B^G$.

<u>CONJ.</u> (Broué, Malle, Rouquier) There exist a Coxeter-like presentation (BMR proved this in all but 6 exceptional cases)

II The Dunkl-De Rham complex

$G \subset GL(V)$ a complex reflection group.

For $H \in \mathcal{A}$ we put $G_H = \{ g \in G \mid g|_H = \mathrm{Id}_H \}$.

Then $G_H \simeq \mathbb{Z}/e_H\mathbb{Z}$ a cyclic group (Steinberg).

and $\widehat{G}_H = \{ \chi_H^{-i} \mid i = 0, \dots, e_H - 1 \}$ $(\chi_H = \det|_{G_H})$

Choose $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$

Thus $\forall g \in G_H : \alpha_H^g = \chi_H^{-1}(g)\alpha_H$

$$\boxed{\begin{array}{l}\underline{\text{Observation}}\ \text{If } p \in \mathbf{P}(V) \text{ and } \forall g \in G_H : \\ p^g = \chi_H^{-i}(g)\,p, \text{ then } p \text{ divisible by } \alpha_H^i \\ (i = 0, \dots, e_H - 1)\end{array}}$$

Let $\boxed{\varepsilon_{H,i} = \frac{1}{e_H} \sum_{g \in G_H} \chi_H^i(g) \cdot g \in \mathbb{C}[G_H]}$

(projection on χ_H^{-i}). Then for $i \neq 0$ we have a well defined linear operator

$$\boxed{\alpha_H^{-1}\varepsilon_{H,i} : P(V) \to P(V)}$$

Choose constants $k_{H,1}, \dots, k_{H,e_H-1} \in \mathbb{C}$ and put:

$$\boxed{a_H(k) = \sum_{i=1}^{e_H-1} e_H\, k_{H,i}\, \varepsilon_{H,i} \in \mathbb{C}[G_H]}$$