Open problems on Cherednik algebras, symplectic reflection algebras, and related topics.

by P. Etingof and V. Ginzburg.

0.1. **Asphérical values.** Let W be a finite irreducible Coxeter group, $\mathfrak{h}$ its reflection representation, and $H_{1,c} = H_{1,c}(W, \mathfrak{h})$ be the rational Cherednik algebra attached to W with parameter c, a conjugation invariant function on the set of reflections in W ([E1], section 7). Let $e = |W|^{-1} \sum_{w \in W} w$ be the symmetrizer for W. Let $B(W)$ be the set of all c for which $H_{1,c} \neq H_{1,c}eH_{1,c}$. This is equivalent to existence of a nonzero module M in category $\mathcal{O}$ over $H_{1,c}$ which is aspherical, i.e. $eM = 0$.

An important open problem is to characterize $B(W)$.

Conjecture 0.1. $c \in B(W)$ if and only if the irreducible representation $L_c(\text{sign})$, whose highest weight is the sign representation of W, is aspherical.

Let $B'(W) \subset \mathbb{C}$ be the intersection of $B(W)$ with the set of constant functions c (If W is simply laced, $B'(W) = B(W)$).

It follows from the paper [DJO] that if $c \in (-1, 0) \cap \mathbb{Q}$, and the denominator of c divides one of the degrees d_i of W, then $c \in B'(W)$.

Conjecture 0.2. If $c \in B'(W)$ then $c \in (-1, 0) \cap \mathbb{Q}$, and the denominator of c divides one of the degrees d_i of W.

This conjecture is known in type A, thanks to [GS] and [BE]. It follows from Conjecture 0.1 thanks to the paper [DJO]. The same paper implies that Conjecture 0.2 also follows from the following weaker conjecture.

Conjecture 0.3. $\forall c \in B'(W)$, $c < 0$.

Finally, it follows from the paper [BE] that to prove Conjecture 0.2, it suffices to show that for $c > 0$, any nonzero finite dimensional $H_{1,c}$-module admits a W-invariant vector.

0.2. **Ring theoretic properties of symplectic reflection algebras.** Let V be a finite dimensional symplectic vector space over $\mathbb{C}$, and G a finite subgroup of $Sp(V)$. Let $H_{1,c}[V,G]$ be the corresponding symplectic reflection algebra (here c is a conjugation invariant function on the set of symplectic reflections).

Conjecture 0.4. The algebra $H_{1,c}[V,G]$ is simple and satisfies Bernstein's inequality (the GK dimension of a nonzero finitely generated module is at least $\frac{1}{2}\dim V$) when c is Weil generic, i.e., does not belong to a countable union of proper subvarieties.

Note that the algebra $H_{1,0}[V,G]$ is simple and satisfies Bernstein's inequality, as it is isomorphic to the smash product of G with the Weyl algebra of V.

This conjecture is known for rational Cherednik algebras ($V = \mathfrak{h} \oplus \mathfrak{h}^*$, G acts on $\mathfrak{h}$), see [BEG], and in the case $\dim V = 2$, due to the work of Crawley-Boevey and Holland [CBH]. However, it is open for symplectic reflection algebras for wreath products.

One may upgrade Conjecture 0.4 to a collection of more refined conjectures and questions. For instance, one may conjecture that the algebra $H_{1,c}[G, V]$, for any c, satisfies the generalized Bernstein inequality, GBI (an algebra A is said to satisfty

GBI is for any nonzero finitely generated module M over A,

$$\mathrm{GKdim}(M) \geq \frac{1}{2}\mathrm{GKdim}(A/\mathrm{ann}(M)),$$

where GKdim is the Gelfand-Kirillov dimension, and ann stands for the annihilator.)

Another question is as follows: let $d(c)$ be the smallest Gelfand-Kirillov dimension of a nonzero finitely generated module over $H_{1,c}[G,V]$ (an integer because this algebra has a filtration whose associated graded is finite over its center); find the stratification of the space of c by values of $d(c)$.

In particular, it is an interesting problem to find c for which $d(c) = 0$, i.e., $H_{1,c}[G,V]$ has finite dimensional representations, and find their G-characters.

In the case when $G = S_n \ltimes \Gamma^n$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$ (wreath product case) and $k := c(s_{ij})$ is "infinitesimally small" (where $s_{ij} \in S_n$ is the transposition of i and j), this problem was solved by S. Montarani and W.L.Gan [Mo], [Ga].

Another interesting conjecture, similar to the Gelfand-Kirillov conjecture for Lie algebras, is the following. Let $e = |G|^{-1} \sum_{g \in G} g$.

Conjecture 0.5. The skew-field of quotients of $eH_{1,c}[G, V]e$ is independent of c, and hence is isomorphic to the skew-field of quotients of $\text{Weyl}(V)^G$, where $\text{Weyl}(V)$ is the Weyl algebra of V.

0.3. **Hecke algebras of orbifolds.** Let X be a simply connected complex manifold, and G a discrete group acting faithfully and holomorphically on X. Then X/G is a complex orbifold.

It is clear that for any $g \in G$, the fixed set X^g is smooth (it is empty unless g has finite order). A reflection hypersurface is a connected component Y of X^g which has codimension 1 in X. If Y is a reflection hypersurface, denote by G_Y the stabilizer in G of a generic point of Y. It is a cyclic group of some order n_Y. Let g_Y be the generator of this group having eigenvalue $e^{2\pi i/n_Y}$ on the normal bundle to Y. Let X' be the set of points $x \in X$ whose stabilizer G_x is trivial. The braid group of the orbifold X/G is the fundamental group $\tilde{G} = \pi_1(X'/G, x)$ for some $x \in X'$. Let C_Y be the conjugacy class in $\tilde{G}$ defined by a small loop around the image of Y in X/G. We have a natural surjective homomorphism $f : \tilde{G} \to G$, and the the kernel K of f is defined by the relation $T^{n_Y} = 1, T \in C_Y$.

Let $\mathbf{t} = (t_{Y,j})$, $j = 1, ..., n_Y$, be a collection of invertible parameters, invariant with respect to conjugation by G. Following [E2] define the orbifold Hecke algebra $\mathcal{H}(\mathbf{t}, X, G)$ over $\mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]$ to be the quotient of the group algebra of the braid group $\mathbb{C}[\tilde{G}][\mathbf{t}, \mathbf{t}^{-1}]$

by the relations

$$\prod_{j=1}^{n_Y}(T - t_{Y,j}e^{2\pi ij/n_Y}) = 0,$$

for $T \in C_Y$.

Thus, the orbifold Hecke algebra is a deformation of the group algebra $\mathbb{C}[G]$. It is proved in [E2] that if $H^2(X,\mathbb{C}) = 0$, then the formal completion of this algebra at the point $t_{Y,j} = 1$ is a *flat* formal deformation of the group algebra $\mathbb{C}[G]$. However, algebraic, rather than formal, flatness is, in most cases, an open problem.

Conjecture 0.6. If $H^2(X,\mathbb{C}) = 0$ then the algebra $\mathcal{H}(\mathbf{t}, X, G)$ is a free $\mathbb{C}[\mathbf{t},\mathbf{t}^{-1}]$-module, which has a basis

$\{T_g, g \in G\}$, specializing to the basis of $\{g, g \in G\}$ at the point $t_{Y,j} = 1$.

In particular, if G has polynomial growth, then $\mathcal{H}(\mathbf{t}, X, G)$ has a finite Gelfand-Kirillov dimension, equal to the GK dimension of G, and if G is finite then $\mathcal{H}(\mathbf{t}, X, G)$ has rank equal to $|G|$.

This conjecture is known only in some special cases. In the finite group case, when X is a vector space, it reduces to the case when G is a complex reflection group, in which case this is the conjecture from [BMR]; it is known in many cases (including all real reflection groups), but not all. The conjecture is also known in the case when W is an affine or double affine Weyl group, and X is the reflection representation (affine and double affine Hecke algebras, [Che]). The conjecture is open in the case when X is a complex vector space and G is a crystallographic reflection group, e.g. $S_n \ltimes (\mathbb{Z}/\ell Z \ltimes \Lambda)^n$, $X = \mathbb{C}^n$, where $\ell = 3, 4, 6$, and Λ is the triangular or square lattice in $\mathbb{C}$.

0.4. **Higher rank analogs of noncommutative surfaces.** Let A be a Calabi-Yau algebra of dimension 2 over $\mathbb{C}$. This means that A has cohomological dimension 2, and $\mathrm{Ext}^i_{A-\mathrm{bimod}}(A, A \otimes A)$ lives in degree 2 and is isomorphic to A as an A-bimodule.

Assume that $HH^0(A) = \mathbb{C}$, $HH^1(A) = 0$. (In this case it is known [EO] that if $n > 1$ then for the algebra $A_n = S_n \ltimes A^n$ one has $HH^2(A_n) = HH^2(A) \oplus \mathbb{C}$, $HH^3(A_n) = 0$. This implies, by classical deformation theory, that the moduli space of formal deformations of A_n is the product of the moduli space of formal deformations of A with a 1-dimensional formal disk. This implies that there exists an interesting 1-parameter (with parameter k) deformation $H_{n,k}(A_u)$ of $S_n \ltimes A_u^{\otimes n}$, where A_u is the universal deformation of A. In fact, one expects the existence of such a deformation even when $HH^1(A) \neq 0$.

An interesting problem, open in many cases, is to explicitly construct this deformation for concrete algebras A, coming from quantized algebraic surfaces, and to give meaning to this deformation for non-formal (i.e., numerical) values of parameters. One expects that the "spherical subalgebra" $eH_{n,k}(A_u)e$ (where $e \in \mathbb{C}[S_n]$ is the Young symmetrizer) will then be a quantization of the Hilbert scheme of the corresponding surface.

To be specific, if A is the Weyl algebra $D(\mathbb{C})$ (quantization of the symplectic plane), then $A_u = A$, and $H_{n,k}(A)$ is the rational Cherednik algebra of type A. If A is the algebra of invariants $D(\mathbb{C})^\Gamma$, $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup, then $H_{n,k}(A_u)$ is a subalgebra in the symplectic reflection algebra for wreath product.

If A is the quantum torus T, then $H_{n,k}(A_u)$ is the double affine Hecke algebra of type A ([Che]). If $A = T^{\mathbb{Z}_2}$, then $H_{k,n}(A_u)$ is a subalgebra in Sahi's 6-parameter algebra [Sa]. If A is the quantization of a Del Pezzo surface with a nodal genus 1 curve removed, described in [EOR], then $H_{k,n}(A_u)$ is the algebra described in [EGO]. However, if A is a quantization of a del Pezzo surface with a smooth elliptic curve removed ([VdB],[EG]), then we don't know how to construct $H_{k,n}(A_u)$. The simplest interesting case is $A = A'/(z - C)$, where A' is a Sklyanin algebra with 3 generators, z its cubic central element, and C a complex number.

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