On Non-Holomorphic Corrections to Black Hole Partition Functions

Gabriel Lopes Cardoso

The Quantum Nature of Space Time
Edinburgh, June 18, 2008

ARNOLD SOMMERFELD
CENTER FOR THEORETICAL PHYSICS

LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

This Talk

This talk is about:

- black hole partition functions for BPS black holes carrying electric/magnetic charges (q, p) in four-dimensional $N = 2$ supergravity theories,
- in particular about duality invariance and non-holomorphic corrections to black hole partition functions.

Work in progress with

Bernard de Wit and Swapna Mahapatra

and based on earlier work with

Bernard de Wit, Jürg Käppeli and Thomas Mohaupt, `hep-th/0601108`.

OSV Proposal

BPS black holes in $N = 2$, e.g. type II Calabi-Yau compactifications.

Define a mixed black hole partition function $Z_{BH}(p, \phi)$ in terms of black hole degeneracies $d(p, q)$ (a suitable index),

$$Z_{BH}(p,\phi) = \sum_q d(p,q)\, \mathrm{e}^{\pi q\,\phi} \quad \xrightarrow{\text{ILPT}} \quad d(p,q) = \int d\phi\, Z_{BH}(p,\phi)\, \mathrm{e}^{-\pi\, q\,\phi}$$

Here, ϕ are the electrostatic potentials. OSV proposal:

`Ooguri + Strominger + Vafa, hep-th/0405146`

$$Z_{BH} \equiv \mathrm{e}^{4\pi\, \mathrm{Im}\, F_{top}} \;,$$

with F_{top} topological free energy. If true, degeneracies computed by the topological string. Weak topological string coupling g_{top}:

$$F_{top}(g_{top}, z^A) = \sum_{g=0}^{\infty} g_{top}^{2g-2}\, F_g(z^A) \quad , \quad \text{holomorphic}$$

IIA: z^A Kähler class moduli of Calabi-Yau threefold.

Modified OSV Proposal

F_g's enter in the Wilsonian action as follows:

- metric on Kähler class moduli space is computed from F_0
- higher F_g's ($g \geq 1$) are coupling functions for higher-curvature terms proportional to the square of the Weyl tensor.

$N = 2$ theory may have duality symmetries. However, OSV proposal does not manifestly respect invariance under dualities.

In addition, duality invariance requires the F_g's ($g \geq 1$) to acquire non-holomorphic corrections:

- needed in the LEEA to make symmetries of the theory manifest;
- encoded in the holomorphic anomaly equations of the topological string.

Therefore, need to change the OSV relation

$$Z_{BH} = \mathrm{e}^{4\pi \,\mathrm{Im}\, F_{top}} \longrightarrow Z_{BH} \propto \mathrm{e}^{4\pi \,\mathrm{Im}\, F_{top}}$$

by a non-trivial proportionality factor (this talk).

Modified OSV Proposal

This non-trivial proportionality factor

- is crucial for maintaining covariance under electric-magnetic duality transformations;
- results in a non-trivial measure factor in the OSV integral for $d(p, q)$ (may also make the integral better convergent).

For large black holes, presence of non-trivial proportionality factor has been established by direct calculation of Z_{BH} in $N = 4$ models,

Shih + Yin, hep-th/0508174
C + de Wit + Käppeli + Mohaupt, hep-th/0601108

Denef + Moore, hep-th/0702146

For small $N = 4$ black holes, the results of Dabholkar + Denef + Moore + Pioline, hep-th/0507014 also indicate need for a measure factor.

Single-Center Black Holes

Further subtlety: does Z_{BH} also count states of multi-center black hole solutions? Microstate degeneracy $d(p,q)$ exhibits jumps due to decay of states at walls of marginal stability, e.g.

Denef + Moore, hep-th/0702146
A. Sen, arXiv:0705.3874

Cheng + E. Verlinde arXiv:0706.2363

Here, will assume that mixed black hole partition function counts only states associated with single-center black holes. See R. Gupta + A. Sen, arXiv:0806.0053 for an argument in this direction.

This Talk

- Work at weak topological string coupling g_{top}.
  See F. Denef + G. Moore, hep-th/0702146 for results at strong topological string coupling.
- Describe a method to include non-holomorphic corrections necessary for duality covariance.
  Suggests consistent non-holomorphic deformation of special geometry. Departure from topological string?
- Use saddle-point arguments to infer measure factor in OSV integral.
- Confront with proposal for microstate degeneracy in a specific $N = 2$ model, the S-T-U model. J. David, arXiv:0711.1971

BPS Black Holes

BPS black holes in $D = 4, N = 2$: extremal, supported by (VM) complex scalar fields Y^I ($I = 0, \ldots, n$), charges (p^I, q_I).

Calabi-Yau compactifications: effective Wilsonian Lagrangian contains higher-curvature interactions $\propto$ Weyl^2 $\rightarrow$ encoded in holomorphic homogeneous function $F(Y, \Upsilon)$. Here Υ is the Weyl background.
Υ-expansion $F = \sum_{g=0}^{\infty} (Y^0)^{2-2g}\, \Upsilon^g\, F_g$.

Attractor mechanism: `Ferrara + Kallosh + Strominger 1995 + 1996`

Also holds in presence of Weyl^2: `C + de Wit + Käppeli + Mohaupt 2000`

At horizon: $Y^I \rightarrow Y^I_{\rm Hor}(p,q)$, $\Upsilon \rightarrow -64$

Attractor equations (in the presence of Weyl^2):

$$
\begin{aligned}
Y^I - \bar{Y}^{\bar{I}} &= i\, p^I \quad , \quad \text{magnetic} \quad , \quad F_I = \partial F(Y, \Upsilon)/\partial Y^I \\
F_I - \bar{F}_{\bar{I}} &= i\, q_I \quad , \quad \text{electric} \quad , \quad F_\Upsilon = \partial F(Y, \Upsilon)/\partial \Upsilon
\end{aligned}
$$

Electro/magnetostatic potentials: $Y^I + \bar{Y}^{\bar{I}}$, $F_I + \bar{F}_{\bar{I}}$

Variational Principle

Attractor equations can be obtained from a variational principle, based on a BPS entropy function Σ: C+de Wit+Käppeli+Mohaupt, hep-th/0601108

$$\Sigma(Y,\bar{Y},p,q) = \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - q_I(Y^I + \bar{Y}^{\bar{I}}) + p^I(F_I + \bar{F}_{\bar{I}}) \,,$$

where $\mathcal{F}$ is the free energy

$$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -i\left(\bar{Y}^{\bar{I}} F_I - Y^I \bar{F}_{\bar{I}}\right) - 2i\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_{\bar{\Upsilon}}\right)$$

Stationary points: set $\Upsilon = -64$

$$\begin{aligned} \delta\Sigma &= i(Y^I - \bar{Y}^{\bar{I}} - ip^I)\,\delta(F_I + \bar{F}_{\bar{I}}) - i(F_I - \bar{F}_{\bar{I}} - iq_I)\,\delta(Y^I + \bar{Y}^{\bar{I}}) \\ \delta\Sigma &= 0 \longleftrightarrow \text{attractor equations} \end{aligned}$$

At attractor point, get macroscopic (Wald's) entropy:
C + de Wit + Mohaupt, hep-th/9812082

$$\pi\Sigma|_{\text{attractor}} = \mathcal{S}_{\text{macro}}(p,q)$$

Duality Transformations

Charges (p^I, q_I) undergo duality transformations. If these constitute symmetries of LEEA, macroscopic entropy is invariant under them.

These leave Υ invariant, but act as symplectic $Sp(2n+2, Z)$ transformations on the vector $(Y^I, F_I(Y, \Upsilon))$ in the attractor equations. Entanglement through the Weyl background!

Precise form of $N = 2$ LEEA with non-holomorphic corrections not known. Thus, rather than relying on an action principle, will use the following procedure: demand that

- attractor equations retain their form;
- they follow from a variational principle based on free energy $\mathcal{F}$.

Consider a general function $F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, not necessarily holomorphic. Associated $\mathcal{F}$ has same form as before,

$$\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = -i \left(\bar{Y}^{\bar{I}} F_I - Y^I \bar{F}_{\bar{I}} \right) - 2i \left(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_{\bar{\Upsilon}} \right)$$

Non-Holomorphic Corrections

Variation of $\mathcal{F}$: with the attractor value $\Upsilon = -64$

$$\begin{aligned} \delta\mathcal{F} &= i(Y^I - \bar{Y}^{\bar{I}})\,\delta(F_I + \bar{F}_{\bar{I}}) - i(F_I - \bar{F}_{\bar{I}})\,\delta(Y^I + \bar{Y}^{\bar{I}}) \\ &\quad - \left[i \left(2\Upsilon\,\delta F_\Upsilon + Y^I\,\delta F_I - F_I \delta Y^I \right) + c.c. \right] \end{aligned}$$

Vanishing of second line: at least two solutions, namely

- F holomorphic, $F(\lambda Y, \lambda^2 \Upsilon^2) = \lambda^2 F(Y, \Upsilon)$, usual Wilsonian case
- $F = 2i\,\Omega$, Ω real, $\Omega(\lambda Y, \lambda \bar{Y}, \lambda^2 \Upsilon, \lambda^2 \bar{\Upsilon}) = \lambda^2\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$

  Without loss of generality:

  $$F = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$$

  When Ω is harmonic (i.e. $\Omega =$ holo + anti-holo), get back usual Wilsonian case.

Consistent Deformation of Special Geometry?

Second option seems to be a consistent non-holomorphic deformation of special geometry, e.g.:

- under symplectic (duality) transformations, $(Y^I, F_I) \rightarrow (\tilde{Y}^I, \tilde{F}_I)$; can show that

$$\tilde{F}_I = \frac{\partial \tilde{F}}{\partial \tilde{Y}^I} \; ;$$

  $F, \tilde{F}$ in same equivalence class;
- additional requirement is that B. de Wit, hep-th/9602060

$$\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_{\bar{\Upsilon}} \longrightarrow \Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_{\bar{\Upsilon}}$$

  transforms as a scalar. It follows that $\mathcal{F}$ and Σ are symplectic scalars, invariant under duality transformations.

  This is the case in a number of examples so far (see later), order by order in an Υ-expansion.

Duality Invariant OSV Integral

Consider the following duality invariant integral, expressed in terms of entropy function Σ, with the attractor value $\Upsilon = -64$,

$$\int d(Y^I + \bar{Y}^{\bar{I}})\, d(F_I + \bar{F}_{\bar{I}})\, \mathrm{e}^{\pi \Sigma(Y,\bar{Y},p,q)} = \int dY\, d\bar{Y}\, \Delta^{-}(Y,\bar{Y})\, \mathrm{e}^{\pi \Sigma(Y,\bar{Y},p,q)}$$

Duality covariance requires measure factor

$$\Delta^{-} = |\, \det \left[\mathrm{Im} \left[F_{JK} - F_{J\bar{K}} \right] \right] |$$

Evaluate integral in saddle-point approximation about attractor point:

$$\mathrm{e}^{\pi \Sigma|_{\text{attractor}}} = \mathrm{e}^{\mathcal{S}_{\text{macro}}(p,q)} \qquad \text{for large charges}$$

Duality invariant. Expect saddle-point approximation to hold for large (dyonic) black holes.

Suggests to identify the above with $d(p,q)$, as in OSV.

Prediction for Mixed Partition Function

On the other hand, when only integrating over $Y^I - \bar{Y}^{\bar{I}}$ in saddle-point approximation so that $Y^I = (\phi^I + ip^I)/2$, get modified OSV-type integral,

$$d(p,q) = \int d\phi \, \sqrt{\Delta^-(p,\phi)} \, \mathrm{e}^{\pi[\mathcal{F}_E(p,\phi) - q_I \phi^I]} \, ,$$

where

$$\mathcal{F}_E = 4 \left[\mathrm{Im} F(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})\right] |_{Y^I = (\phi^I + ip^I)/2 \quad , \quad \Upsilon = -64}$$

Inverting yields prediction for $N = 2$ mixed black hole partition function,

$$Z_{BH}(p,\phi) = \sum_q d(p,q) \, \mathrm{e}^{\pi q_I \phi^I} = \sqrt{\Delta^-} \, \mathrm{e}^{\pi \mathcal{F}_E} = \sqrt{\Delta^-} \, \mathrm{e}^{4\pi \Omega^{\text{nonholo}}} \, \mathrm{e}^{\pi \mathcal{F}_E^{\text{holo}}}$$

Test requires knowledge of microscropic degeneracies.

Tests in $N = 4$

Microscopic degeneracies known for a class of $N = 4$ models, e.g. the so-called CHL models (rank ≤ 28).
Υ-dependence of Ω severely restricted.

`Dijkgraaf + Verlinde + Verlinde, hepth/9607026`

`D. Jatkar + A. Sen, hep-th/0510147`

These models can be treated in an $N = 2$ setup, with eight additional charges associated with the four additional graviphotons of $N = 4$:

- Procedure outline above for incorporating non-holomorphic terms into the BPS entropy function Σ is consistent with the asymptotic degeneracy, $d(p,q) = \exp \mathcal{S}_{\rm macro}$.

  `C+de Wit+Käppeli+Mohaupt,hep-th/0412287;Jatkar+Sen, hep-th/0510147`

- Direct calculation of mixed partition function Z_{BH} establishes presence of proportionality factor; agrees with $\sqrt{\Delta^-}\, e^{4\pi\Omega^{\rm nonholo}}$.

  `Shih+Yin, hep-th/0508174, C+de Wit+Käppeli+Mohaupt, hep-th/0601108`

$N = 2$: scarce. One example: S-T-U-model, `J. David, arXiv:0711.1971`

Class of $N = 2$ Models

Specific class of $N = 2$ models with exact duality symmetry groups:

- exact classical moduli spaces, parametrized by

$$S = -i\frac{Y^1}{Y^0} \quad , \quad T^a = -i\frac{Y^a}{Y^0} \quad , \quad a = 2, \ldots, n$$

- with $F = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ given by

$$F^{(0)}(Y) = i\left(Y^0\right)^2 \, S\, T^a\, \eta_{ab}\, T^b \quad , \quad \eta_{ab} = \text{const} \quad , \quad \chi = 0$$

- assume S-duality (or subgroup of $SL(2, Z)$)

$$S \rightarrow \frac{aS - ib}{icS + d} \quad , \quad ad - bc = 1$$

  and similarly for T-duality.
- Two such models:

  FHSV-model `Ferrara+Harvey+Strominger+Vafa, hep-th/9505162`

  S-T-U-model `A. Sen + Vafa, hep-th/9508064`

Constraints on Ω

S-duality represented by a symplectic transformation on (Y^I, F_I).
Υ unaffected. However, T^a NOT invariant, S-T-Υ-mixing,

$$T^a \longrightarrow T^a + \frac{ic}{\Delta_S\,\left(Y^0\right)^2}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b} \quad , \quad \Delta_S = icS + d$$

Different from what is done in the topological string context.
Requiring transformation of Y^I to correctly induce transformation of F_I upon substitution yields:

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial T^a}\right)' &= \frac{\partial\Omega}{\partial T^a} \\
\left(\frac{\partial\Omega}{\partial S}\right)' - {\Delta_S}^2\,\frac{\partial\Omega}{\partial S} &= \frac{\partial({\Delta_S}^2)}{\partial S}\left[-\tfrac{1}{2}Y^0\frac{\partial\Omega}{\partial Y^0} - \frac{ic}{4\,\Delta_S\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\frac{\partial\Omega}{\partial T^b}\right] \\
\left(Y^0\frac{\partial\Omega}{\partial Y^0}\right)' &= Y^0\frac{\partial\Omega}{\partial Y^0} + \frac{ic}{\Delta_S\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\frac{\partial\Omega}{\partial T^b}
\end{aligned}$$

Similarly for T-duality transformations.

Consequences for Ω

Expand in powers of Υ (with $\Upsilon = \bar{\Upsilon}$),

$$\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = \sum_{g=1}^{\infty} \Omega^{(g)}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \quad , \quad \Omega^{(g)} \propto \Upsilon^{g} ,$$

assume that duality invariance is realized order-by-order in Υ.

- At order Υ, $\Omega^{(1)}$ coincides with the solution to the holomorphic anomaly equation of the topological string.
- At higher order, have departure from topological string due to S-T-Υ-mixing. Lower $\Omega^{(g)}$ transform into higher $\Omega^{(g)}$.
- $\Upsilon\, \partial_\Upsilon F = 2i\, \Upsilon\, \partial_\Upsilon \Omega$ is invariant.

Consequences for Ω

For the FHSV-model:

$$\Omega^{(2)} = -\frac{G_2(2S)}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b} - \frac{1}{4(Y^0)^2}\,\frac{\partial \ln\Phi(T)}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b}\,\frac{\partial\Omega^{(1)}}{\partial S} + \mathrm{c.c}$$

Related to, but different from: $G_2(S) \to \hat{G}_2(S,\bar{S}) = G_2(S) - 12/(S+\bar{S})$

T. Grimm + A. Klemm + M. Mariño + M. Weiss, arXiv:hep-th/0702187

Both approaches describe the same deformation, but encoded in a different way. Precise relationship remains to be worked out, proceeds through relationship with real special geometry (Hesse potential).

Prediction for Mixed Partition Function

Recall prediction for mixed partition function,

$$Z_{BH}(p,\phi) = \sum_q d(p,q)\, \mathrm{e}^{\pi q_I \phi^I} = \sqrt{\Delta^-(p,\phi)}\, \mathrm{e}^{\pi \mathcal{F}_E(p,\phi)}$$

$$\mathcal{F}_E = 4\left[\mathrm{Im}F(Y,\bar Y,\Upsilon,\bar\Upsilon) - \Omega(Y,\bar Y,\Upsilon,\bar\Upsilon)\right]\big|_{Y^I=(\phi^I+ip^I)/2\,,\ \Upsilon=-64}$$

For the class of $N=2$ models discussed above (which have $\chi = 0$), and restricting to $\Omega = \Omega^{(1)}$ in first approximation (topological string), get

$$\sqrt{\Delta^-}\,\mathrm{e}^{4\pi\Omega^{(1)\mathrm{nonholo}}} = \mathrm{e}^K \longrightarrow Z_{BH} = \mathrm{e}^{\pi\,\mathcal{F}_E^{\mathrm{holo}}}\,\mathrm{e}^K\ ,$$

where K is Kähler potential $K = -\log\left[(S+\bar S)(T+\bar T)^a\,\eta_{ab}\,(T+\bar T)^b\right]$.
Different from strong-coupling prediction of D & M, hep-th/0702146
Compare with $N=4$ CHL-models: $\sqrt{\Delta^-}\,\mathrm{e}^{4\pi\Omega^{(1)\mathrm{nonholo}}} = \mathrm{e}^{-K}$

Shih+Yin,hep-th/0508174; C+de Wit+Käppeli+Mohaupt,hep-th/0601108; Denef+Moore,hep-th/0702146

Qualitative difference between $N = 8,4,2$ at weak coupling g_{top}!
Confront this with microscopic proposal for the $N=2$ S-T-U-model.

S-T-U-Model:

- 3 vector moduli S, T, U; $\Omega^{(1)} = \Omega^{(1)}(S, T, U, \bar{S}, \bar{T}, \bar{U})$;
  `A. Sen + Vafa, hep-th/9508064;`
  `Gregori + Kounnas + Petropoulos, hep-th/9901117`
- exact duality symmetry $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$,
  with $\Gamma(2) \subset SL(2, Z)$, i.e. $a, d \in 2Z + 1$, $b, c \in 2Z$

Proposal for microscopic degeneracy for twisted sector dyons:

`J. David, arXiv:0711.1971`

Degeneracy captured by the zeros of a Siegel modular form Φ_0 of weight zero of (a subgroup of) the genus 2 modular group $Sp(2, Z)$:

$$I(K, L, M) = \oint d\rho d\sigma dv \, \frac{e^{i\pi[\rho K + \sigma L + (2v-1)M]}}{\Phi_0(\rho, \sigma, v)} .$$

The degeneracy proposal uses 9 charge bilinears K, L, M (8 charges),

$$d_{\mathrm{STU}}(p, q) = I(K_S, L_S, M_S)\, I(K_T, L_T, M_T)\, I(K_U, L_U, M_U) .$$

For large black holes, large charges: asymptotic degeneracy reproduces macroscopic entropy. `J. David`

S-T-U-model

However... and unlike $N = 4$,

- saddle-point equations do not quite agree with the attractor equations...
- direct evaluation of the mixed partition function does not appear to result in $Z_{BH} \propto \exp[\pi \mathcal{F}_E]$ ...
- can construct small black holes with charges $(q_2, q_3; p^0)$.
  5D black hole in Taub-NUT space.
  3 out of 9 charge bilinears non-vanishing. Non-holomorphic corrections in $\Omega^{(1)}(S, T, U, \bar{S}, \bar{T}, \bar{U})$ crucial for stabilizing the moduli:

$$S_{\rm Hor} = \frac{1}{2}\sqrt{L_S} \quad , \quad T_{\rm Hor} = 2\frac{\sqrt{L_S}}{K_T} \quad , \quad U_{\rm Hor} = 2\frac{\sqrt{L_S}}{K_U} \; .$$

Macroscopic entropy: $\mathcal{S}_{\rm macro} = \pi\sqrt{L_S} + \log - {\rm terms}$

How to extract this from $d_{\rm STU}$? Suggests that better understanding of $d_{\rm STU}$ is needed.

Conclusions

- We made a proposal for a measure factor in the OSV-integral that is compatible with duality covariance in $N = 2$ models.
- In doing so, we proposed a method for incorporating non-holomorphic terms needed for duality invariance. We found indications that this is a consistent non-holomorphic deformation of special geometry.

  We checked this explicitly in the context of the FHSV- and the STU-model, where we also explicitly computed $\Omega^{(2)}$.
- In $N = 4$ models, our proposal yields results in agreement with direct calculations of Z_{BH}.
- We are currently attempting to check our proposal in the $N = 2$ S-T-U-model.

Thanks!