Topological strings, matrix models and emergent geometry

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Introduction

In this talk I will review how the spacetime geometry of certain (topological) string theories emerges in a precise way from large N matrix models. I will also discuss the problem of strong/weak interpolation for the 't Hooft coupling in this context.

In this respect, what I will discuss can be regarded as a toy model of AdS/CFT –albeit in some ways more complicated and subtle than other toy models proposed in the same vein.

What is topological string theory?

Topological string theory can be regarded as a toy model of a full-fledged string theory. Its starting point is a *2d topological field theory*, the topological sigma model, with action

$$S = \frac{1}{\ell_s^2} \int \mathrm{d}z \sqrt{g}\, g^{\mu\nu}\, G_{I\overline{J}} \partial_\mu \phi^I \partial_\nu \phi^{\overline{J}} + \cdots$$

The *target* of this sigma model is a *Calabi–Yau threefold* X. We recall that this is a six-dimensional, Kähler manifold which satisfies Einstein's vacuum equations

$$R_{IJ} = 0.$$

The A and the B models

As in usual string theory, we can study the target metric $G_{I\overline{J}}$ by perturbing the 2d action with graviton vertex operators. However, due to the topological nature of the theory, only a limited set of fluctuations can be incorporated. It is known that for a CY

$$\mathcal{M}_{\text{metrics}} = \mathcal{M}_{\text{Kahler}} \times \mathcal{M}_{\text{complex}}$$

There is a version of topological string theory (called the A model) which incorporates $\mathcal{M}_{\text{Kahler}}$ ("sizes"), while the B model incorporates $\mathcal{M}_{\text{complex}}$ ("shapes"). We will then study generating functions for perturbations of the metric

$$F_g^A(t_{\text{Kahler}}) = \langle \mathrm{e}^{-S(t_{\text{ref}})+t_{\text{Kahler}}V_{\text{Kahler}}} \rangle_{\Sigma_g},$$
$$F_g^B(t_{\text{complex}}) = \langle \mathrm{e}^{-S(t_{\text{ref}})+t_{\text{complex}}V_{\text{complex}}} \rangle_{\Sigma_g}.$$

Mirror symmetry

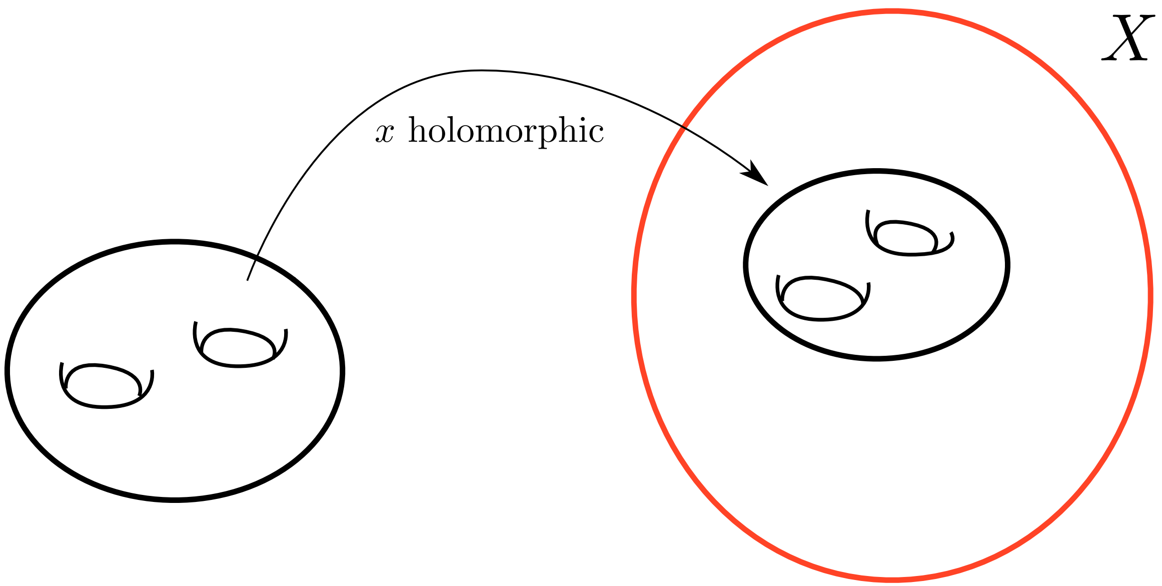
These two perturbations are in fact related, since there is a big duality in the space of CYs which is called *mirror symmetry*. This postulates that, given a CY X, there is another CY $\tilde{X}$ such that

$$F_g^A(t_{\mathsf{Kahler}}; X) = F_g^B(t_{\mathsf{complex}}; \tilde{X})$$

where t_{Kahler} and t_{complex} are related by the so-called *mirror map*.

The free energies

In the A model the free energy at genus g is exhausted by a semiclassical expansion around *instantons* of the 2d sigma model. These are *holomorphic maps* from a Riemann surface of genus g to the Calabi–Yau X



Due to the topological nature of the theory, only one-loop fluctuations around the instantons survive, and the expansion reads

$$F_g(t) = \sum_{d \geq 1} N_{d,g} \mathrm{e}^{-dt/\ell_s^2},$$

where d is the degree of the instanton and t is the Kähler parameter which specifies the size of the embedded worldsheet. The *total* free energy is given by

$$F(t, g_s) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}.$$

The quantum parameters

This theory has *two quantum parameters*:

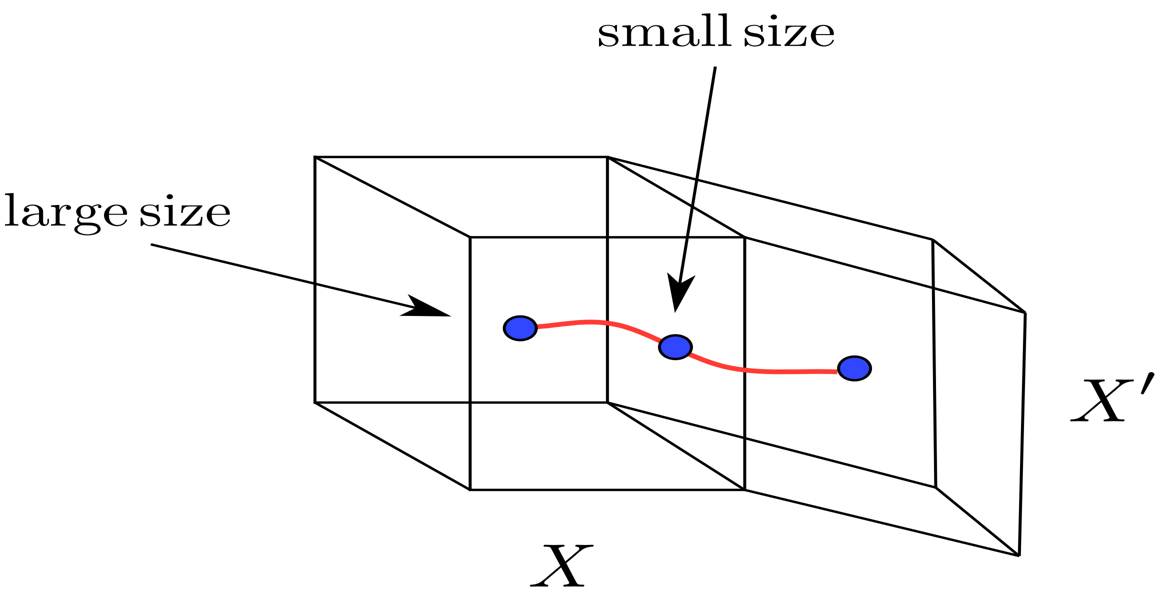
- In the worldsheet, $\hbar_{\mathrm{ws}} = \ell_s^2$, and strong coupling is *small size* $t \sim \ell_s^2$.

- In spacetime, $\hbar_{\mathrm{st}} = g_s$.

The semiclassical expansion above is only valid for t, g_s small. But in many cases one can go to strong coupling in both variables.

Lessons from nonperturbative effects in $\hbar_{\mathsf{ws}}$

For example, by incorporating all effects in $\hbar_{\mathsf{ws}}$ we can move to small values of t and observe in this way space-time topology change [Aspinwall–Greene–Morrison, Witten].



Large N dualities

The best way to understand the inclusion of $\hbar_{\rm st}$ is to use large N dualities. These were first described for topological strings by Gopakumar and Vafa in 1998 and relate topological string theory on certain CY backgrounds to gauge theories in the $1/N$ expansion, in the spirit of the AdS/CFT correspondence.

It turns out that, in the cases where the duality is known to exist, the gauge theory side reduces essentially to a *matrix model*. This is good news, since it allows us to describe in detail the $1/N$ expansion.

Matrix models

We recall that matrix models are defined by matrix integrals of Hermitian $N \times N$ matrices

$$F = \log \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}\, V(M)}$$

This free energy has an asymptotic expansion around $g_s = 0$ of the form

$$F = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}, \qquad t = N g_s$$

where t, the 't Hooft parameter, is kept fixed.

Master fields

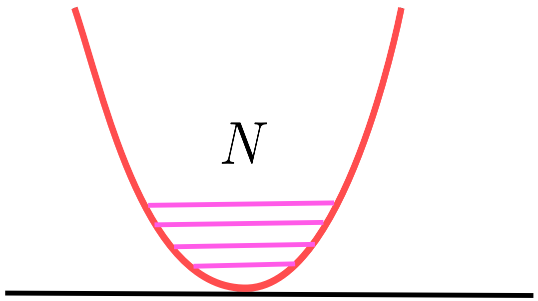
This is a system in which the master field describing the planar limit can be described in detail. First, one can diagonalize M and rewrite the model in terms of N eigenvalues

$$M \to \lambda_1, \cdots, \lambda_N.$$

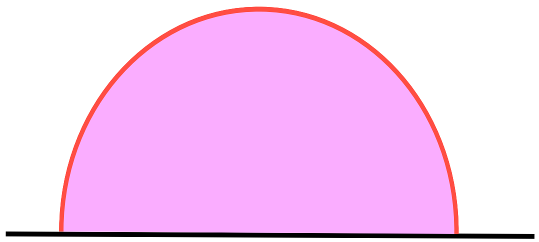
in a potential

$$V_{\text{eff}}(\lambda_i) = V(\lambda_i) - g_s \sum_{j \neq i} \log(\lambda_i - \lambda_j)^2$$

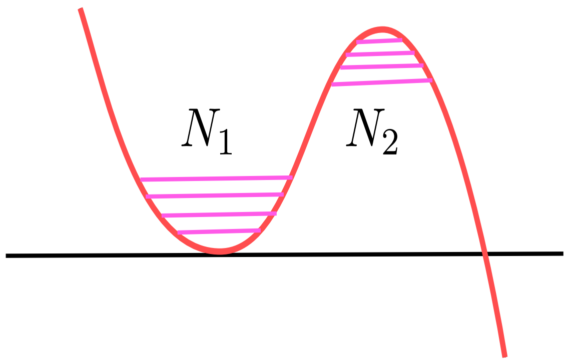
In the large N limit, the λ_i reach an equilibrium distribution and they sit along intervals in the complex plane. Their density at large N is described by a function $\rho(\lambda)$ supported on these intervals. In general we have $N \to N_1 + N_2 + \cdots$ and partial 't Hooft couplings $t_i = g_s N_i$

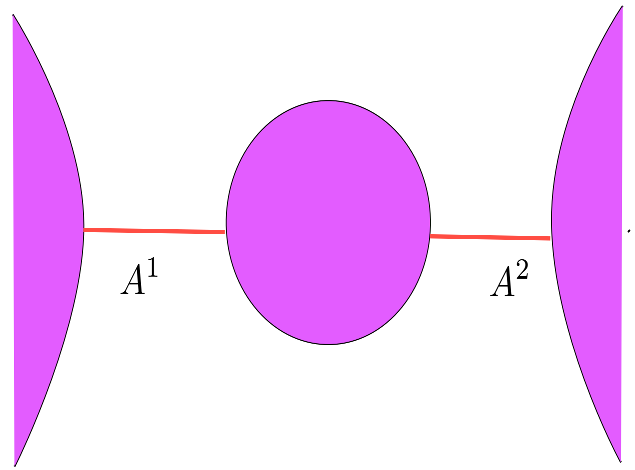


$$V(x) = \frac{x^2}{2}$$



$$\rho(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}$$





Equivalently, the intervals can be regarded as *branch cuts* of a *classical spectral curve* $y = y(\lambda)$ characterizing the model. The density of eigenvalues is given by

$$\rho(\lambda) = \frac{1}{2\pi t} \mathrm{Im}\, y(\lambda).$$

$1/N$ **expansion**

Given the spectral curve, all free energies $F_g(t_i)$ and correlators can be computed explicitly [Ambjorn et al, Eynard]. For example, the planar free energy is

$$\frac{\partial F_0}{\partial t} = \int_B y(z)\mathrm{d}z.$$

where B is a cycle going to infinity. There are explicit recursion relations to compute the rest of the info *from properties of the spectral curve alone*

Gauge/string theory dictionary

The gauge/string dictionary implies in particular

$$F_g^{\mathrm{MM}}(t) = F_g^{\mathrm{TS}}(t_{\mathrm{Kahler}})$$

and we identify $t^i_{\mathrm{Kahler}} = g_s N_i$.

Notice that, as in the AdS/CFT correspondence, the *large radius point* is mapped to strong 't Hooft coupling, while weak coupling is *small distances* in the CY. Difficult, but tractable (and lessons to be learned!).

A simple example: the resolved conifold

The simplest example of this picture is the Calabi–Yau known as *resolved conifold*

$$L \oplus L^{-1} \to \mathbb{S}^2$$

where L is the monopole bundle of charge 1 over $\mathbb{S}^2$. The space of Kähler deformations of the metric is just $t \in \mathbb{C}$, which describes the (complexified) size of the $\mathbb{S}^2$.

As $t \to 0$ we have a simple example of topology change (the *flop transition*).

$U(N)$ gauge theory: Chern–Simons/matrix model

Topological string theory on this space should be equivalent to $U(N)$ *Chern–Simons theory* on $\mathbb{S}^3$ in the large N expansion [Gopakumar–Vafa]. This is an exactly solvable gauge theory [Witten] whose action is

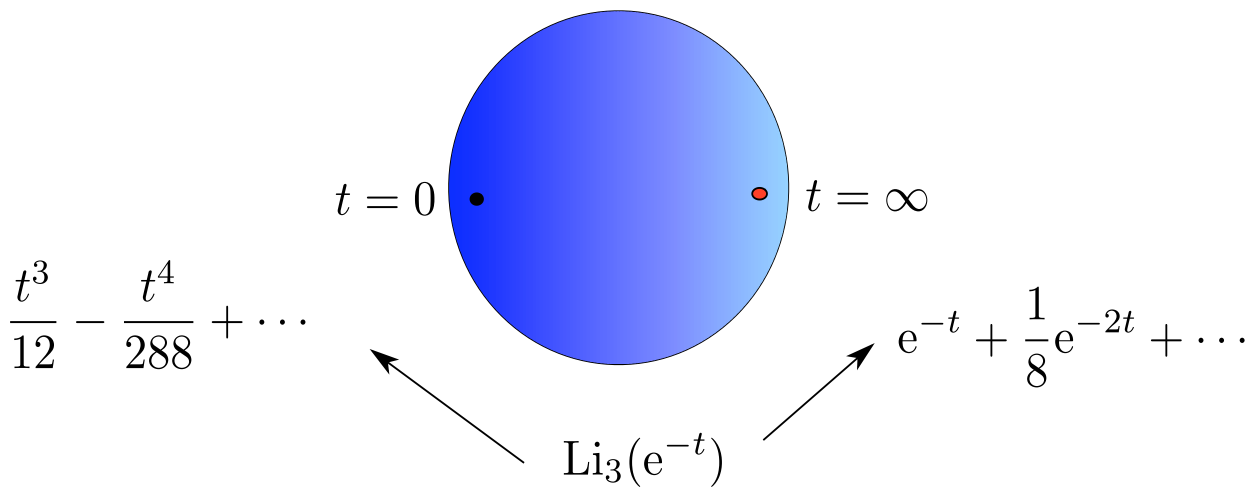
$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge)$$

This theory is in turn equivalent to a "non-standard" matrix model [M.M], which effectively has [Tierz]

$$V(M) = (\log M)^2$$

Strong/weak interpolation: a "trivial" case

In this case it can be obtained by expanding a single function in two different regimes. For $F_0(t)$ one has



Emergent geometry from large N matrix model

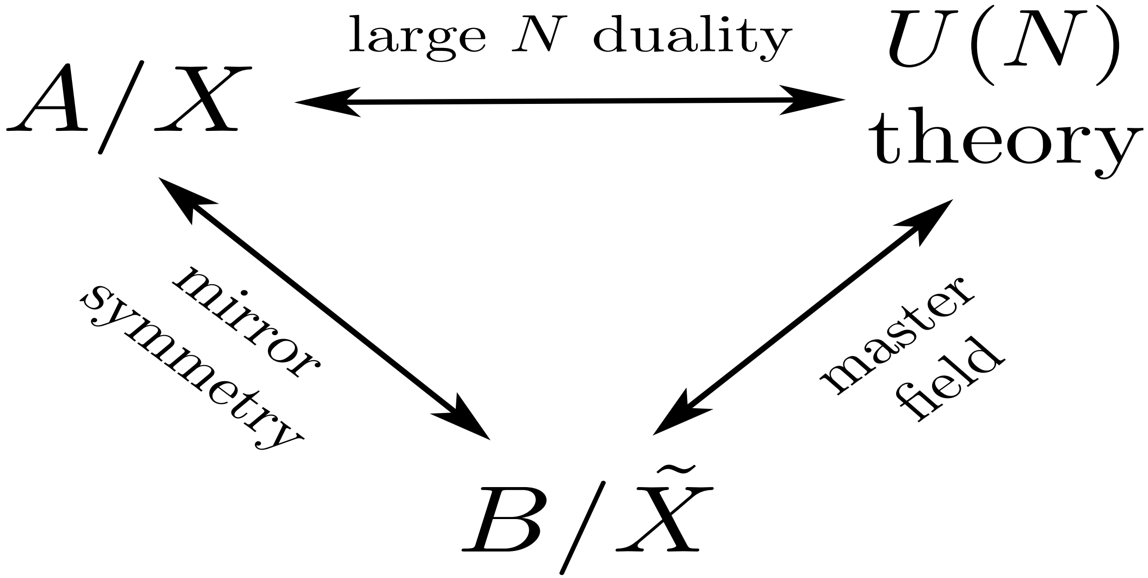
This MM has *one* cut and the spectral curve can be explicitly computed

$$y(p) = \log \left[\frac{\sqrt{(1+p)^2 - 4 z_t p}}{1+p} \right], \qquad z_t = \mathrm{e}^{-t},$$

where $t = N g_s$. This curve has a geometric meaning: it describes the *mirror geometry* to the original CY background!

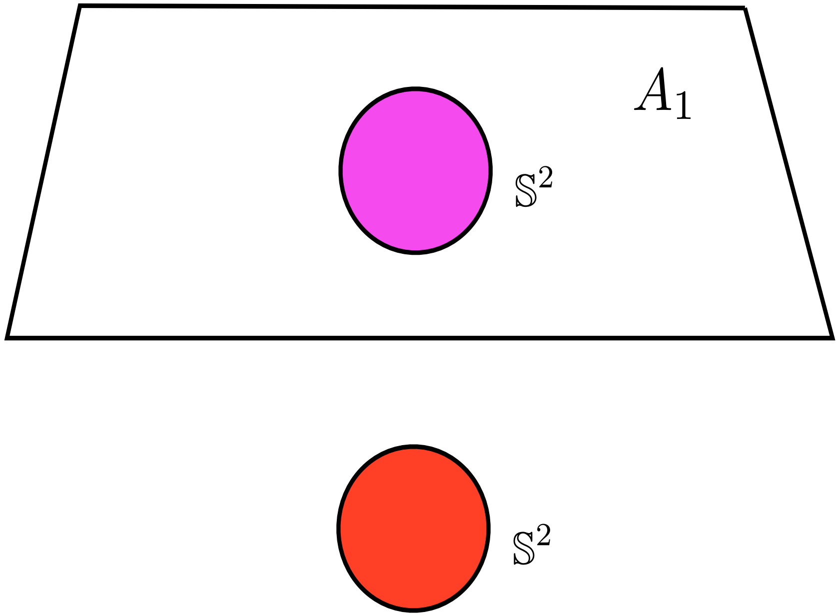
Summary

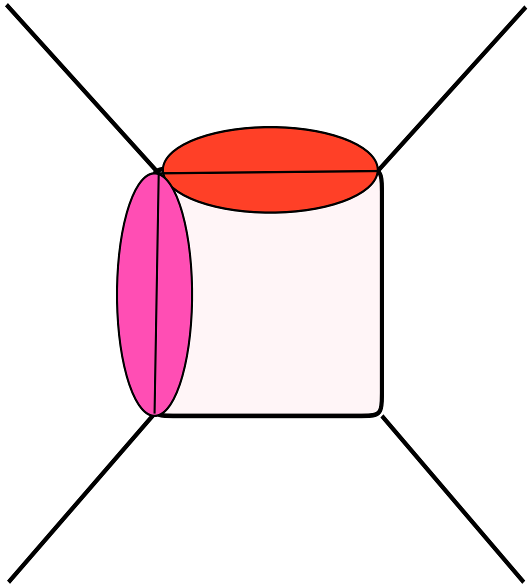
Therefore, the *master field* of the $U(N)$ gauge theory *reconstructs* the *full target geometry* and makes *mirror symmetry* manifest



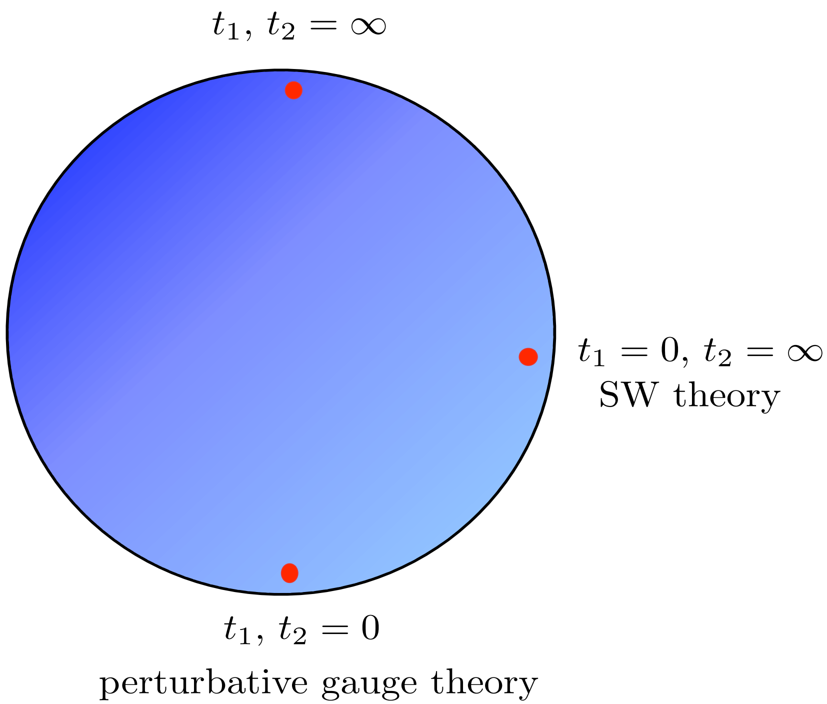
A more complicated example

A more complicated CY: $\mathbb{F}_0$ = fibration of the A_1 4d singularity over $\mathbb{S}^2$. It has *two* Kähler deformations t_1, t_2 (size of the base $\mathbb{S}^2 + \mathbb{S}^2$ in the resolved singularity).





The moduli space has (complex) dimension two, and various points where one can make contact with other important models



Common aspects

- It has a large N dual (CS theory on $\mathbb{S}^3/\mathbb{Z}_2$) with a matrix model description [Aganagic-Klemm-M.M.-Vafa].

- This time we need *two* cuts, since we have two parameters. The master field/spectral curve reconstructs the mirror geometry [Halmagyi et al., BKMP]

New features

The $F_g(t_1, t_2)$ computed at different points are *not* related by analytic continuation due to a nontrivial *modularity structure*. For instance,

$$F_1(t_1, t_2) = \log \eta(\tau(t_1, t_2))) \to \log \eta(-1/\tau) = F_1 + (\log(-\mathrm{i}\tau))/2$$

In particular, *strong/weak coupling interpolation needs a non-trivial quasimodular transformation* [Aganagic et al., BKMP]

The *full* gauge theory partition function involves a *sum over topological sectors*

$$Z_{\rm CS}(N, g_s) = \sum_{N_1+N_2=N} Z_{\rm TS}(N_1, N_2)$$

This is in fact a sum over *(spacetime) instantons* in the matrix model/topological string [M.M.]

A general correspondence

For so-called *toric* Calabi–Yau manifolds, one can always regard the mirror geometry as a spectral curve of a matrix model, and use MM techniques to compute string amplitudes [Bouchard-Klemm-M.M.-Pasquetti], including *nonperturbative* effects $\mathcal{O}(\mathrm{e}^{-1/g_s})$ [M.M.- Schiappa-Weiss, M.M.].

But: It is not clear that one can write a matrix integral underlying all these systems i.e. the theory of "quantum spectral curves" is *a priori* more general than matrix integrals [Eynard–Orantin].

Other pictures of emergent geometry for topological strings

- Spectral curves describing the target spacetime of topological string theory arise also from sums over 2d [Caporaso et al., Eynard] or 3d partitions ("melting crystals") [Okounkov et al., Iqbal et al.]

- There is a generalization of the picture of bubbling geometries to the topological string setting [Gomis–Okuda].