Instanton Corrections and Black Holes Partition Functions

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Introduction

- Explaining the microscopic origin of the thermodynamical properties of black holes is a pass/fail test for any theory of quantum gravity. String theory has passed this test with great success for BPS BH in the strict limit $Q = \infty$. How about $1/Q^n$ corrections, or even $Q = \mathcal{O}(1)$?
- Asking detailed questions of this kind may lead to new connections with advanced mathematics, which will remain even if String Theory was a TOE-bN: enumerative geometry, automorphic forms, number theory,...
- I will argue that precision counting of BPS BH in 4D is closely related to studying instanton corrections to certain couplings in 3D, a problem of physical interest in its own right.

Outline

- Solitons in D dimensions vs instantons in $D-1$
  D=3 gauge theories, D-instantons vs D-branes, ...
- Black holes vs instantons in $D=4, \mathcal{N}=2$ SUGRA
  GW vs GV vs DT, OSV conjecture, ...
- Instanton corrections to hypermultiplet moduli space
  Linear perturbations of QK manifolds, Kontsevich Soibelman,...

Based on work with Gunaydin, Neitzke, Vandoren, Waldron [hep-th/0512296, 0607227,0701214,0707.0267] and work in progress with Alexandrov, Saueressig, Vandoren.

Instanton physics in D-dimensions and soliton spectrum in $D+1$ dimensions are oftentimes related:

- The instantons responsible for confinement of compact QED in $D = 2 + 1$ are obtained by dimensional reduction of the 't Hooft-Polyakov magnetic monopole in $D = 3 + 1$.

Polyakov

- Some aspects of $D = 3 + 1$ $\mathcal{N} = 2$ SYM can be studied by considering its reduction on a circle. This is a non-linear sigma model on a HK space $\mathcal{M}_3$, a torus bundle over the 4D moduli space. Dyons in $D = 3 + 1$ generate instanton corrections to the metric on $\mathcal{M}_3$.

Seiberg Witten

D-branes vs. D-instantons

D-instanton corrections to R^4 couplings in $D=9$ type II string theories are directly related to the D0-brane spectrum in $D=9+1$ type IIA:

- the Witten index of the $U(N)$ Matrix QM splits into "bulk" and "boundary" contributions,

$$\mathrm{Tr}(-1)^F e^{-\beta H}|_{\beta\to\infty} = \mathrm{Tr}(-1)^F e^{-\beta H}|_{\beta\to 0} + \int_0^\infty d\beta \frac{\partial}{\partial\beta} \mathrm{Tr}(-1)^F e^{-\beta H}|_{\beta\to\infty}$$

Yi; Sethi Stern

- The "bulk" part is non-zero due to the continuous part of the spectrum. The "boundary" contribution $\beta\to 0$ reproduces the D-instanton measure, as a consequence of T-duality:

$$1 = \left(1 + \sum_{d|N, d<N} \frac{1}{d^2}\right) - \sum_{d|N, d<N} \frac{1}{d^2} \quad : \quad \Omega(N) \sim \mu(N)$$

Green Gutperle

Black holes vs. world-sheet instantons I

Topological amplitudes $\sum F_g(t) R_+^2 F_+^{2g-2}$ in $\mathcal{N}=2$ $D=4$ SUGRA are determined by the black hole spectrum in $D=5$:

- One-loop contributions of 5D BH in M/CY in a graviphoton background yield all ws instanton contributions in type IIA/CY:

$$F(t^i, \lambda) - F_{\rm polar} = \sum_{Q_i, d, r, m} n_Q^r \frac{1}{m} \left(2 \sinh \frac{m\lambda}{2} \right)^{2r-2} e^{2\pi i Q_i t^i}$$

- the GV invariants n_Q^r are related to indexed degeneracies of BMPV BHs in 5D, realized as M2-branes wrapping 2-cycles:

$$\Omega_5(Q, J_L^3) = (-)^{r+1} \sum_r \binom{2r+2}{r+1+2J_L^3} n_Q^r \quad \sim e^{\sqrt{Q^3-(J_L^3)^2}}$$

Gopakumar Vafa; Katz Klemm Vafa;Huang Klemm Marino Tavanfar

D-dim Black hole entropy and D-dim effective action I

- The Bekenstein-Hawking-Wald formula relates the effective action to the number of BH micro-states (provided the thermo ensemble is correctly identified):

$$S_{BHW} = 2\pi \int \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{g}\, d^2\sigma$$

- For BPS black holes in $\mathcal{N} = 2$ SUGRA, in the presence of the infinite set of topological amplitudes $\sum F_g(t) R_+^2 F_+^{2g-2}$,

$$S_{BHW}(p^I, q_I) = \langle \pi \mathrm{Im} F(p^I + i\phi^I) - q_I \phi^I \rangle$$

Cardoso de Wit Mohaupt

suggestive of the OSV conjecture

$$\Omega(p^I, q_I) = \int d\phi^I \, |Z_{\rm top}(p^I + i\phi^I)|^2 e^{-q_I \phi^I}$$

D-dim Black hole entropy and D-dim effective action II

- This potentially relates worldsheet instantons in IIA/CY to 4D black hole degeneracies. However $\Omega(p, q)$, while locally constant in t^i, jumps at lines of marginal stability. Is there a formula for $\Omega(p^I, q_I; t^i)$ at any point t^i in moduli space ?
- The asymmetric treatment of electric and magnetic charges raises concern about electric-magnetic duality non-manifest.

Cardoso de Wit Mohaupt

- As I'll try to advocate, instanton corrections to certain terms in the 3D effective action may be the right framework to address these issues.

4D BH spectrum and instanton corrections in 3D I

- In $D = 4$, $\mathcal{N} = 2$ SUGRA with n_V vector multiplets and n_H hypers, the moduli space splits into a product of a projective Kähler manifold and a quaternionic-Kähler manifold,

$$\mathcal{M}_V^{(2n_v)} \times \mathcal{M}_H^{(4n_H)}$$

- Upon reducing along (Euclideanized) time, $\mathcal{M}_H$ goes along for the ride, while $\mathcal{M}_V$ is extended to a second QK manifold

$$\mathcal{M} = \mathbb{R}^+_U \times \mathcal{M}_V \times T^{2n_v+2}_{\zeta^I,\tilde{\zeta}_I} \times S^1_\sigma \qquad \equiv \mathrm{c-map}(\mathcal{M}_V)$$

where e^U is the radius of the time direction, $\zeta^I, \tilde{\zeta}_I$ are the electric and magnetic Wilson lines, and σ is the NUT scalar, dual to the KK connection. The σ circle is non-trivially fibered over T^{2n_v+2}.

4D BH spectrum and instanton corrections in 3D II

- At tree level, the 4D hypermultiplet space $\mathcal{M}_H$ is also given by the c-map construction from the v.m. moduli space $\mathcal{M}_{\tilde{V}}$ of the T-dual string theory on the same CY. There is also a one-loop correction $\propto \chi(X)$, and probably no higher order perturbative corrections.

Cecotti Ferrara Girardello; Antoniadis Minasian Theisen Vanhove; Robles-Llana, Saueressig, Vandoren

- The QK metric on $\mathcal{M}_H$ receives instanton corrections from from D2-branes wrapped on $\Gamma \in H_3(X)$, and from k NS5-branes on X. The contributions of D2-branes wrapped on A-cycles can be computed using type IIB S-duality.

Becker Becker Strominger; Robles Llana Saueressig Vandoren

- Similarly, the QK metric on $\mathcal{M}$ receives instanton corrections from 4D black holes with charge $\Gamma \in H_{\rm even}(X)$, as well as geometries with non-trivial NUT charge along the time direction.

4D Black holes and BPS geodesics I

- Spherically symmetric, stationary, extremal black holes fall in the ansatz

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U}(dr^2 + r^2 d\Omega_2^2)$$

Eom imply that $U, t^i, \bar{t}^i, \zeta^I, \tilde{\zeta}_I, \sigma$ as functions of the inverse radial distance $\rho = 1/r$ trace out a null geodesic on $\mathcal{M}^*$, the analytic continuation of $\mathcal{M}$ under $(\zeta, \tilde{\zeta}) \to (i\zeta, i\tilde{\zeta})$. The electric q_I, magnetic p_I and NUT charge k correspond to the momenta on $T^{2n_v+2} \times S^1$.

- BPS black holes correspond to special geodesic, such that at each point,

$$\exists \epsilon^{A'} / p_{AA'} \epsilon^{A'} = 0 \quad \text{i.e.} \quad \epsilon^{A'B'} p_{AA'} p_{BB'} = 0$$

Here $p_{AA'}$ is the momentum in the "quaternionic viel-bein" basis, afforded by the restricted holonomy $SU(2) \times USp(2n_v+2)$ of $\mathcal{M}_3$.

- For vanishing k, this reproduces the attractor flow equations

$$dU/d\rho = -e^U |Z| \,, \quad dt^i/d\rho = -2e^U g_{i\bar{j}} \partial_{\bar{j}} |Z|$$

where $Z = e^{K/2}(q_I X^I - p^I F_I)$ is the central charge.

- The flow eqs can be recast in Hamilton-Jacobi form as

$$p_a = \partial_{\phi^a} S \,, \quad S_{p,q} = e^U |Z_{p,q}| + p^I \tilde{\zeta}_I - q_I \zeta^I$$

- Semi-classically, the radial wave function for a BPS BH with charges p^I, q_I is therefore $\Psi_{p,q} \sim e^{iS_{p,q}}$. Quantum mechanically, the equations $(\epsilon^{A'B'} \nabla_{AA'} \nabla_{BB'} + \kappa \Sigma_{AB})\Psi = 0$ are solved by

$$\Psi_{p,q} = e^{-2U} J_0 \left(e^U |Z_{p,q}| \right) e^{i(p^I \tilde{\zeta}_I - q_I \zeta^I)}$$

Ooguri Vafa Verlinde; Neitzke BP Vandoren

BPS instantons and BPS geodesics I

- Similarly, spherically symmetric instantons in D-dim SUGRA are of the form

$$ds_D^2 = e^{f(r)}(dr^2 + r^2 d\Omega_{D-1}^2)$$

The eom require that $U, t^i, \zeta^I, \tilde{\zeta}_I, \sigma$, as a function of r^α, trace out a null geodesic in $\mathcal{M}^{**}$, the analytic continuation $\mathrm{Re}(t^i) \to i\mathrm{Re}(t^i)$, $\tilde{\zeta}_I \to i\tilde{\zeta}_I, \sigma \to i\sigma$ of $\mathcal{M}$.

- Supersymmetry requires the same condition as for BPS BH's,

$$\exists \epsilon^{A'} / p_{AA'}\epsilon^{A'} = 0 \quad \text{i.e.} \quad \epsilon^{A'B'} p_{AA'} p_{BB'} = 0$$

The hypermultiplets are attracted to fixed values at the waist $r \to 0$ of the wormhole.

Bernhdt, Gaida, Luest, Mohapatra, Mohaupt; Gutperle Spalinski

BPS instantons and BPS geodesics II

- The classical action of the instanton is therefore an analytic continuation of the WKB phase of the BH radial wave function, e.g. for $k = 0$

$$S_{p,q} = e^U |Z_{p,q}| + i(p^I \tilde{\zeta}_I - q_I \zeta^I)$$

  The real part agrees with (length of circle)×(mass of 4D BH). Instantons will contribute like e^{-S}. How about loop corrections in the instanton background ?

- If we could identify some coupling in the 3D action governed by the same equation as for the BH wave function, namely $(\epsilon^{A'B'} \nabla_{AA'} \nabla_{BB'} + \kappa \Sigma_{AB})\Psi = 0$, we would be in business !

Instanton corrections to the hypermultiplet metric I

- Instanton corrections to the QK metric can be encoded in a single function χ, the hyperkähler potential on $\mathcal{S}$, the Swann bundle (or HKC) of $\mathcal{M}$:

$$\mathbb{C}^2\backslash\{(0,0)\} \to \mathcal{S} \to \mathcal{M}$$

  $\mathcal{S}$ is a hyperkähler manifold with an isometric $SU(2)$ action and homothetic Killing vector.

- Perturbations of χ around a given solution precisely satisfy this equation ! As shown by Lebrun, deformations of a QK manifold $\mathcal{M}$ are governed by $H1(Z, \mathcal{O}(2))$, where $Z = \mathcal{S}//U(1)$ is the twistor space of $\mathcal{M}$:

$$\delta\chi = e^{-2U} \oint \frac{dz}{2\pi i z} \Phi \left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

  For $\Phi = e^{i(p_I \xi^I - \tilde{\xi}_I q^I)}$, one recovers the BH wave function !

HK manifolds and symplectomorphisms I

- The HK $\mathcal{S}$ manifold carries an S^2 worth of complex structures,

$$J = \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} J^1 + \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} J^2 + \frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} J^3$$

- With respect to this J,

$$\Omega = (\omega_1 + i\omega_2) + \zeta\omega_3 - (\omega_1 - i\omega_2)\zeta^2$$

  is a holomorphic symplectic form (twisted by $\mathcal{O}(2)$).

- Locally, around $\zeta = \zeta_i$, there exist complex Darboux coordinates such that $\Omega = d\nu_i^I \wedge d\mu_{Ii}$.

Hitchin, Karlhede, Lindstrom, Rocek

HK manifolds and symplectomorphisms II

- On the overlap of two patches, (ν_i^I, μ_{Ii}) and $(\nu_j^I\wedge, \mu_{Ij})$ must be related by a complex symplectomorphism, generated by some function $S(\nu_i, \mu_j)$,

$$\nu_j = \partial_{\mu_j} S(\nu_i, \mu_j)\ , \quad \mu_i = \partial_{\nu_i} S(\nu_i, \mu_j) \quad (*)$$

  The HKC property requires that $S(\nu_i, \mu_j)$ is homogeneous of degree 1 in ν_i^I, and without explicit dependence on ζ.
- When $\nu_i^I(\zeta), \mu_{Ii}(\zeta)$ are constrained to be Taylor series regular at 0, the space of solutions to (*) is finite dimensional, and equal to the HK space $\mathcal{S}$ itself. Its Kahler form is obtained by Taylor expanding Ω to first order. The metric on the QK space $\mathcal{M}$ is obtained by the standard superconformal quotient.

Hitchin, Karlhede, Lindstrom, Rocek; de Wit, Rocek, Vandoren

$\mathcal{O}(2)$ hyperkähler cones

- For 4n-dim HKC with n triholomorphic isometries, the ν^I's can be chosen to be the globally defined ($\mathcal{O}(2)$ twisted) moment maps, so the transition functions take the form

$$\nu_j = \nu_i \; , \quad \mu_i = \mu_j + \partial_{\nu_i} H(\nu_i) \; , \quad S(\nu_i, \mu_j) = \nu_i \mu_j + H(\nu_i)$$

  $H(\nu_I)$ is homogeneous function of degree 1, known as the "generalized prepotential" .

- the HK metric follows from the tensor Lagrangian

$$\mathcal{L}(v^L, \bar{v}^L, x^L) = \oint \frac{d\zeta}{2\pi i} H(\nu^L(\zeta)) \; , \quad \nu^L = v^L + x^L \zeta^L - \bar{v}^L \zeta^2$$

  by Legendre transform,

$$\langle \chi(v^L, \bar{v}^L, w_L + \bar{w}_L) + x^L(w_L + \bar{w}_L) \rangle = \mathcal{L}(v^L, \bar{v}^L, x^L)$$

- For $\mathcal{S} = HKC(\mathrm{c-map}(\mathcal{M}_V))$, $H(\nu^L) = F_0(\nu^I)/\nu^\sharp$

Rocek Vafa Vandoren

Perturbations around $\mathcal{O}(2)$ spaces

- General deformations of $\mathcal{O}(2)$ spaces can be described by perturbing the complex symplectomorphisms,

$$S(\nu_i, \mu_j) = \nu_i \mu_j + H(\nu_i) + H^{(1)}(\nu_i, \mu_j)$$

  and working out the deformed twistor lines to linear order in $H^{(1)}$.
- After performing the superconformal quotient, one finds that, in accordance with Lebrun's theorem,

$$\delta\chi = e^{-2U} \oint \frac{dz}{2\pi i z} H^{(1)} \left[1, \xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

Abelian and non-Abelian Fourier coefficients I

- The Fourier expansion of $\delta\chi$ along the twisted torus T^{2n_v+3} is

$$\delta\chi = \sum_{p,q\in\mathbb{Z}^{2n_v+2}} \chi_{p,q}(U,t^i,\bar{t}^i)\, e^{i(q_I\zeta^I - p^I\tilde{\zeta}_I)} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \chi_k\, e^{ik\sigma}$$

- The Abelian Fourier coefs $\chi_{p,q}(U,t^i,\bar{t}^i)$ are expected to encode (the boundary contribution to) 4D BH indexed degeneracies at given point in $\mathcal{M}$. If indeed $\chi_{p,q}(U,t^i,\bar{t}^i) \sim \Omega(p,q)$, the sum is severely divergent, and needs to be regularized.
- The non-Abelian coefs χ_k carry info about non-physical BH with non-trivial NUT charge along time direction. When $k\neq 0$, p^I and q_I are no longer well-defined ! Rather, one should use an appropriate basis of Landau wave functions on T^{2n_v+2} threaded with magnetic field $k d\zeta^I \wedge d\tilde{\zeta}_I$.

Abelian and non-Abelian Fourier coefficients II

- Beyond the linear approximation, one should consider finite symplectomorphisms of the twisted torus T^{2n_v+3}. The infinite instanton sum should be replaced by a (non-commutative) product of symplectomorphisms. This seems to be the right physical set-up to make sense of Kontsevich-Soibelman wall crossing formula

$$\prod_{\arg(Z_{p,q}) asc.} T_{p,q}^{\Omega_+(p,q)} = \prod_{\arg(Z_{p,q}) desc.} T_{p,q}^{\Omega_-(p,q)}$$

courtesy of Gaiotto Moore Neitzke, in progress

  If so, this would support $\chi_{p,q} \sim \Omega(p,q)$, and possibly resolve the issue of divergences.

- Eventually, one hopes that modular invariances fixes all Fourier coefficients.

Conclusion

- We have suggested to study 4D BH degeneracies via their contributions to couplings in the 3D effective action. For $\mathcal{N} = 2$ BH, this is equivalent to computing instanton corrections to the hypermultiplet metric. Can we go beyond leading order ?
- Just as the vector multiplet branch receives higher-derivative F-term corrections $F(\lambda, t^i)$, the hypermultiplet branch receives higher-derivative corrections $\tilde{F}(S, \lambda, t^i)$, which now depend on the string coupling S. $\Psi_{\rm gen} = e^{\tilde{F}}$ should be a natural one-parameter generalization of $\Psi_{\rm top}$. Can it be computed ?
- For $\mathcal{N} = 8$, $\mathcal{N} = 4$ (and magic $\mathcal{N} = 2$) theories, this suggests that BPS BH degeneracies could be obtained as Fourier coefficients of the 3D duality group $SO(8, n_v + 2)$ and $E_{8(8)}$. Relation to DVV-type genus 2 partition functions ?