The Fluid-Gravity correspondence

Mukund Rangamani

Durham University

Gravitational Thermodynamics and the Quantum Nature of Spacetime, ICMS, Edinburgh.
June 18, 2008

Bhattacharyya, Hubeny, Minwalla, MR

Bhattacharyya, Hubeny, Loganayagam, Mandal, Minwalla, Morita, MR, Reall

Motivation

Understanding gravity in higher dimensions

What is the class of classical black hole solutions in dimensions $D > 4$?

- Inhomogeneous black holes?
- Systematic technique to construct solutions?

Validity of semi-classical gravity

In AdS/CFT, which class of field theory states have a nice semi-classical gravitational description?

- Understanding the emergence of geometry.
- What are “legal” singularities in gravity (cosmic censorship)?

Outline

Outline

References:

- Significant work over the last 6 years in this subject pioneered by

  Policastro, Son, Starinets

  Herzog, Kovtun, Buchel

  many, many others . . .

- Inspiration for our work

  Janik, Peschanski

  Bhattacharyya, Lahiri, Loganayagam, Minwalla

- Related very interesting recent developments

  Baier, Romatschke, Son, Starinets, Stephanov

  Loganayagam

  Van Raamsdonk

  Bhattacharyya, Loganayagam, Minwalla, Nampuri, Trivedi, Wadia

Fluid dynamics

What is fluid dynamics?

- Fluid dynamics is continuum low energy description of any microscopic QFT valid when scales of variation are large compared to mean free path ℓ_{mfp}.
- The fluid description assumes that the system achieves local thermodynamic equilibrium.

Regime of validity

If the local temperature in the fluid is T and the scales of variation of the dynamical degrees of freedom are L, local equilibration demands

$$L\,T \equiv \frac{1}{\epsilon} \gg 1$$

Fluid dynamics

Dynamical degress of freedom:

- Local temperature T.
- Fluid velocity u_μ (normalized $\eta^{\mu\nu} u_\mu u_\nu = -1$).
- Particle and charge densities ρ and q_i.
- Pressure P and chemical potentials determined by equation of state.
- For conformal fluids $P = \frac{1}{d-1} \rho$, and we will set all charges to zero.

Conformal Fluid dynamics

The stress tensor

Encode all the fluid information into an energy momentum tensor $T^{\mu\nu}$, which is

- Traceless: $T^\mu_\mu = 0$
- Conserved: $\nabla_\mu T^\mu_\nu = 0$.

The conservation equation encapsulates the dynamical content of fluid dynamics.

Conformal Fluid dynamics

The stress tensor

Encode all the fluid information into an energy momentum tensor $T^{\mu\nu}$, which is

- Traceless: $T^{\mu}_{\mu} = 0$
- Conserved: $\nabla_{\mu} T^{\mu}_{\nu} = 0$.

Conformal fluid stress tensor

$$T^{\mu\nu} = \alpha\, T^d \left(\eta^{\mu\nu} + d\, u^{\mu}\, u^{\nu}\right) + \pi^{\mu\nu}$$

where

- we've used the equation of state for conformal fluids.
- $\pi^{\mu\nu}$ is the dissipative part of the stress-tensor.
- Work in Landau gauge: $\pi^{\mu\nu} u_{\nu} = 0$.

Validity of semi-classical gravity

Gravity dual to field theory

- Given a boundary stress tensor $T^{\mu\nu}$ we can construct an asymptotically AdS solution.
- The stress tensor is related to the normalizable modes of the gravitational field in AdS.

$$ds^2 = \frac{dz^2 + \left(\eta_{\mu\nu} + \alpha\, z^d\, T_{\mu\nu}\right)\, dw^\mu\, dw^\nu}{z^2}$$

Degrees of freedom counting

- A boundary conformally invariant stress tensor has $\frac{d(d+1)}{2} - 1$ degrees of freedom.
- **Q**: Can any such stress tensor give a regular bulk geometry?

Validity of semi-classical gravity

Gravity dual to field theory

- Given a boundary stress tensor $T^{\mu\nu}$ we can construct an asymptotically AdS solution.
- The stress tensor is related to the normalizable modes of the gravitational field in AdS.

Degrees of freedom counting

- A boundary conformally invariant stress tensor has $\frac{d(d+1)}{2} - 1$ degrees of freedom.
- **Q**: Can any such stress tensor give a regular bulk geometry?

Validity of semi-classical gravity

Regularity and dof truncation

- Claim: Regular solutions is given by stress tensors that are fluid dynamical.
- For pure gravity + cosmological constant, this is a reduction of degress of freedom, since fluid stress tensors have d dof, viz., T and u_μ.
- Uniqueness: In fact, the gravity solutions thus constructed are the most general regular long-wavelength solutions to Einstein's equations (gravity & -ve cc).

The Schwarzschild-AdS black hole

Consider the well-known stationary solution; the boosted Schwarzschild-AdS black hole.

▸ Schwarzschild-AdS metric

- It is parameterized by d parameters;

$$\text{horizon size}: \;\; b = \frac{1}{\pi T}$$

$$\text{boost parameters}: \;\; \beta_i \rightsquigarrow u^\mu$$

- In a fluid picture the black hole is dual to a perfect fluid with

$$T^{\mu\nu} = \frac{1}{b^4} \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right)$$

The Schwarzschild-AdS black hole

- We now promote u_μ and b to fields on the boundary: $u_\mu(x)$ and $b(x)$
- Call such a metric $g^{(0)}$ – it does not satisfy the equations of motion.

$$E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R - 6\, g_{MN} = 0$$

- Starting from here we will construct an iterative solution.

A perturbation scheme for gravity

- Assume that the variation in local temperature and velocities are slow
$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}(\epsilon) \ , \qquad \frac{\partial_\mu u}{T} \sim \mathcal{O}(\epsilon)$$
- In local patches the solution is like a boosted black brane.
- The perturbative scheme is aimed at constructing a regular bulk solution, by patching together pieces of the uniform boosted brane.

A perturbation scheme for gravity

We will construct solutions in a derivative expansion with ϵ as a book-keeping parameter. The metric and the dynamical variables b and u_μ get corrected order by order

$$g = \sum_{k=0}^{\infty} \epsilon^k \, g^{(k)}$$

$$b = \sum_{k=0}^{\infty} \epsilon^k \, b^{(k)}$$

$$u = \sum_{k=0}^{\infty} \epsilon^k \, u^{(k)}$$

Our task is to determine $g^{(k)}$, etc..

A perturbation scheme for gravity

At a given order in the ϵ-expansion we find equations for $g^{(k)}$. These are ultra-local in the field theory directions and take the schematic form:

$$\mathbb{H}\left[g^{(0)}(u_\mu^{(0)}, b^{(0)})\right] g^{(k)}(x^\mu) = s_k$$

- $\mathbb{H}$ is a second order linear differential operator in r alone.
- s_k are regular source terms which are built out of $g^{(n)}$ with $n \leq k-1$.

▸ Computational details

A perturbation scheme for gravity

Importantly the equations of motion split up into two kinds:

- Constraint equations: $E_{r\mu} = 0$, which implement stress-tensor conservation (at one lower order).
- Dynamical equations: $E_{\mu\nu} = 0$ and $E_{rr} = 0$ allow determination of $g^{(k)}$.

We solve the dynamical equations

$$g^{(k)} = \text{particular}(s_k) + \text{homogeneous}(\mathbb{H})$$

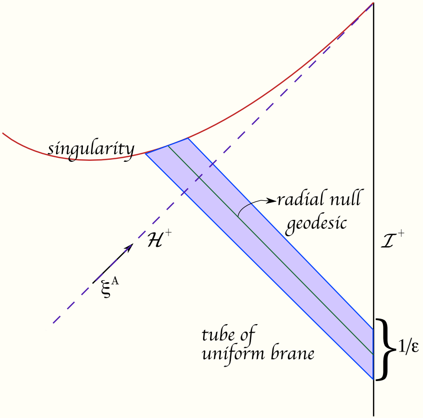
with boundary conditions (i) regularity in the interior (ii) asymptotically AdS.

First order computation

Explicit solution

The spacetime geometry dual to fluids

The bulk solution thus constructed is tubewise approximated by a black brane!



The causal structure of bulk solution illustrating the tubular domains where the geometry is that of an uniformly boosted Schwarzschild-AdS black hole.

The event horizon

- The background has a regular event horizon.
- One can determine the event horizon locally using the fact that the solution settles down at late times to an uniformly boosted black brane.
- The horizon location can be determined within the perturbation scheme

$$r = r_H(x) = \frac{1}{b(x)} + \sum_{k=1}^{\infty} \epsilon^k \, r_{(k)}(x)$$

- In fact, $r_{(k)}(x)$ is determined algebraically by demanding that the surface given by $r = r_H(x)$ be null.

The Entropy current

- Given a bulk geometry with a horizon we can determine the Bekenstein-Hawking entropy.
- Bulk construction of entropy: using area-form a of spatial slices of the event horizon in Planck units.

Fluid entropy current

The area-form on event horizon can be pulled back to the boundary to define a fluid entropy current J_S^μ

$$J_S = *_\eta a$$

with non-negative divergence

$$\partial_\mu J_S^\mu \geq 0$$

▸ Entropy properties ▸ Entropy details

The 4-dimensional conformal fluid from AdS_5

The dissipative part of the stress tensor $\pi^{\mu\nu}$ has a gradient expansion that can be explicitly determined from our construction of the bulk solution.

- At first order, we have the shear term, whose coefficient is the shear-viscosity η.
- At second order, we have 5 possible operators which transform Weyl covariantly.

$$T^{\mu\nu}_{(2)} = \tau_\pi \, \eta \, \mathcal{T}_1^{\mu\nu} + \kappa \, \mathcal{T}_2^{\mu\nu} + \lambda_1 \, \mathcal{T}_3^{\mu\nu} + \lambda_2 \, \mathcal{T}_4^{\mu\nu} + \lambda_3 \, \mathcal{T}_5^{\mu\nu}$$

Baier, Romatschke, Son, Starinets, Stephanov

Loganayagam

The 4-dimensional conformal fluid from AdS_5

The $\mathcal{N} = 4$ Super-Yang Mills fluid: The fluid parameters are

$$\eta = \frac{N^2}{8\pi} \, (\pi T)^3$$

$$\tau_\pi = \frac{2 - \ln 2}{\pi T} \, , \qquad \lambda_1 = \frac{2\,\eta}{\pi T} \, ,$$

$$\lambda_2 = \frac{2\,\eta \, \ln 2}{\pi T} \, , \qquad \lambda_3 = 0.$$

which agrees with the results of Baier, Romatschke, Son, Starinets, Stephanov. They also derive the curvature coupling term:

$$\kappa = \frac{\eta}{\pi T}$$

▸ 2^{nd} order $T_{\mu\nu}$

Summary of salient points

Gravitational results

- Constructed a map between gravity solutions with regular horizons and hydrodynamic configurations.
- Given any fluid configuration, algorithmic method to construct the bulk geometry order by order in a derivative expansion.

Fluid predictions

- Prediction of fluid parameters: recover well known value for η and predict second order coefficients.
- Entropy current with non-negative divergence constructed directly from bulk geometry.

Open questions

Lessons for Gravity

- Is there qualitative difference between gravitational dynamics between $2+1$ and $3+1$ dimensions? Van Rammsdonk
- Cosmic censorship for fluid duals.
- Phase structure of solutions with exotic horizon topology.
- Extremal fluid duals.
- Relation to the membrane paradigm: “The membrane at the end of the universe”.

Fluid Lessons

- Is graviational entropy current special?
- Turbulence?

The Schwarzschild-AdS black hole metric

Useful form of the Schwarzschild-AdS metric

$$ds^2 = -2\, u_\mu\, dx^\mu dr - r^2\, f(b\, r)\, u_\mu\, u_\nu\, dx^\mu dx^\nu + r^2\, P_{\mu\nu}\, dx^\mu dx^\nu \ ,$$

with

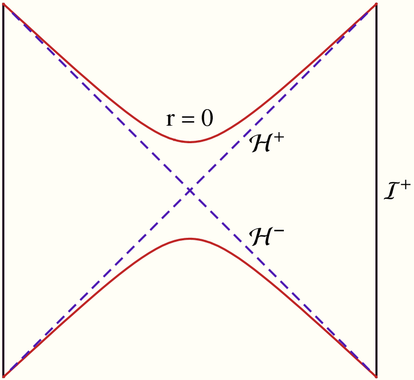
$$f(r) = 1 - \frac{1}{r^4}$$

$$u_v = -\frac{1}{\sqrt{1-\beta^2}} \qquad u_i = \frac{\beta_i}{\sqrt{1-\beta^2}} \ ,$$

$$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu\, u_\nu$$

Uniform brane solution

The Schwarzschild-AdS black hole metric



The causal structure of the boosted Schwarzschild-AdS black hole

◂ Uniform brane solution

Some details of the computation

Coordinate chart: For dealing with regularity issues etc., it is simplest to work in an analog of ingoing Eddington-Finkelstein coordinates.

$$ds^2 = -2\, u_\mu(x)\, \mathcal{S}(r,x)\, dr\, dx^\mu + \chi_{\mu\nu}(r,x)\, dx^\mu\, dx^\nu$$

- The choice of coordinates is such that $x^\mu = \textit{constant}$ are ingoing null geodesics.
- It is well adapted to discuss features of horizon, such as entropy in the fluid language.
- The metric functions have an ϵ-expansion in our perturbation scheme.

Some details of the computation

The operator $\mathbb{H}$: Useful to decompose metric perturbations into $SO(3)$ representations: scalars **1**, vectors **3** and symmetric traceless tensors **5**. For instance, we find:

$$\mathbb{H}_{\mathbf{3}}\# = \frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr}\#\right)$$

$$\mathbb{H}_{\mathbf{5}}\# = \frac{d}{dr}\left(r^5\, f(r)\,\frac{d}{dr}\#\right)$$

The source terms: These differ at various orders in perturbation theory. At first order:

$$s_1^{\mathbf{3}} = -\frac{3}{r^2}\,\partial_v\,\beta_i^{(0)}$$

$$s_1^{\mathbf{5}} = -6\,r^2\,\sigma_{ij}^{(0)}$$

Perturbation scheme

Computation at first order

Details of first order computation:

- To solve the equations to first order we need to ensure conservation of the perfect fluid stress tensor

$$\partial_\mu T^{\mu\nu}_{(0)} = 0$$

  which needs to be solved only locally (at say $x^\mu = 0$).

- This can be used to eliminate derivatives of $b^{(0)}$ in terms of those of $\beta_i^{(0)}$.

$$\partial_v b^{(0)} = \frac{1}{3}\,\partial_i\,\beta_i^{(0)}\,, \qquad \partial_i\,b^{(0)} = \partial_v \beta_i^{(0)}$$

- Then we solve $\mathbb{H} g^{(1)} = s_1$ where the operators and sources are given as follows:

Computation at first order

Details of first order computation:

- For vectors **3** and symmetric traceless tensors **5** of $SO(3)$, we find:

$$\mathbb{H}_{\mathbf{3}}\# = \frac{d}{dr}\left(\frac{1}{r^3}\,\frac{d}{dr}\#\right)$$

$$\mathbb{H}_{\mathbf{5}}\# = \frac{d}{dr}\left(r^5\, f(r)\,\frac{d}{dr}\#\right)$$

- The corresponding source terms are

$$s_1^{\mathbf{3}} = -\frac{3}{r^2}\,\partial_v\,\beta_i^{(0)}$$

$$s_1^{\mathbf{5}} = -6\,r^2\,\sigma_{ij}^{(0)}$$

Perturbation scheme

Explicit solution to second order

$$ds^2 = -2\, u_\mu\, dx^\mu dr - r^2\, f(b\, r)\, u_\mu u_\nu\, dx^\mu dx^\nu + r^2\, P_{\mu\nu}\, dx^\mu dx^\nu + 3\, b^2\, h_2(b\, r)\, u_\mu\, dx^\mu dr + \mathcal{G}_{\mu\nu}\, dx^\mu dx^\nu$$

$$\mathcal{G}_{\mu\nu} = r^2\, \left(2\, b\, F(b\, r)\, \sigma_{\mu\nu} + b^2\, \alpha^{(2)}_{\mu\nu}(b\, r)\right) + \frac{1}{r^2}\, \left(\frac{2}{3}\, r^3\, \partial_\lambda u^\lambda\, u_\mu\, u_\nu + \frac{k_2(b\, r)}{b^2}\, u_\mu\, u_\nu\right) + r^2\, b^2\, h_2(b\, r)\, P_{\mu\nu} + \frac{1}{r^2}\, \left(-2\, r^3\, \mathcal{D}u_\alpha + \frac{1}{b^2}\, j^{(2)}_\alpha(b\, r)\right)\, P^\alpha_\nu\, u_\mu\, .$$

Back to perturbation scheme

Properties of Entropy current

- The bulk-boundary pull-back is facilitated by our coordinate chart – pull-back along radial ingoing geodesics.

$$x^\mu(\mathcal{H}) \quad \longrightarrow \quad x^\mu(\text{bdy})$$

- Fluid entropy current consistent with second law and equations of motion has a 5 parameter ambiguity.
- Bulk construction of entropy current is ambiguous, but less so:
  (i) ability to add total derivative terms without changing area
  (ii) pull-back is ambiguous to boundary diffeomorphisms.
  At second order this results in a two parameter ambiguity for Weyl covariant current with positive divergence.

◂ Entropy current

Expression for entropy current

The gravitational entropy current:

$$(4\pi\,\eta)^{-1}\,J_S^\mu = \left[1 + b^2\left(A_1\,\sigma_{\alpha\beta}\,\sigma^{\alpha\beta} + A_2\,\omega_{\alpha\beta}\,\omega^{\alpha\beta} + A_3\,\mathcal{R}\right)\right]u^\mu + b^2\left[B_1\,\mathcal{D}_\lambda\sigma^{\mu\lambda} + B_2\,\mathcal{D}_\lambda\omega^{\mu\lambda}\right] + C_1\;b\,\ell^\mu + C_2\;b^2 u^\lambda\,\mathcal{D}_\lambda\ell^\mu + \ldots$$

with

$$A_1 = \frac{1}{4} + \frac{\pi}{16} + \frac{\ln 2}{4}; \qquad A_2 = -\frac{1}{8}; \qquad A_3 = \frac{1}{8}$$

$$B_1 = \frac{1}{4}; \qquad B_2 = \frac{1}{2}$$

$$C_1 = C_2 = 0$$

Entropy current

Expression for entropy current

Non-negativity of divergence: $\mathcal{D}_\mu J_S^\mu \geq 0$ demands

$$B_1 = 2\,A_3\;, \qquad C_1 + C_2 = 0$$

which follows from the fact that

$$4\,G_N^{(5)}\,b^3\,\mathcal{D}_\mu J_S^\mu = \frac{b}{2}\left[\sigma_{\mu\nu} + b\left(2\,A_1 + 4\,A_3 - \frac{1}{2} + \frac{1}{4}\ln 2\right)u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} + 4\,b\,(A_2 + A_3)\,\omega^{\mu\alpha}\omega_\alpha{}^\nu + b\,(4\,A_3 - \frac{1}{2})(\sigma^{\mu\alpha}\,\sigma_\alpha{}^\nu) + b\,C_2\,\mathcal{D}^\mu \ell^\nu\right]^2 + (B_1 - 2A_3)\,b^2\,\mathcal{D}_\mu \mathcal{D}_\lambda \sigma^{\mu\lambda} + (C_1 + C_2)\,b^2\,\ell_\mu \mathcal{D}_\lambda \sigma^{\mu\lambda} + \ldots$$

◂ Entropy current

The 4-dimensional conformal fluid from AdS_5

The second order stress tensor

$$\left(16\,\pi\, G_N^{(5)}\right)\; T^{\mu\nu} = (\pi\, T)^4\,(\eta^{\mu\nu} + 4\, u^\mu u^\nu) - 2\,(\pi\, T)^3\,\sigma^{\mu\nu} + T^{\mu\nu}_{(2)}$$

$$\sigma^{\mu\nu} = P^{\mu\alpha}\, P^{\nu\beta}\;\partial_{(\alpha} u_{\beta)} - \frac{1}{3}\, P^{\mu\nu}\,\partial_\alpha u^\alpha$$

is the shear of the fluid and

$$P^{\mu\nu} = \eta^{\mu\nu} + u^\mu\, u^\nu$$

is a co-moving spatial projector and finally

$$T^{\mu\nu}_{(2)} = (\pi T)^2\left((\ln 2)\; T^{\mu\nu}_{2a} + 2\, T^{\mu\nu}_{2b} + (2 - \ln 2)\left[\frac{1}{3}\, T^{\mu\nu}_{2c} + T^{\mu\nu}_{2d} + T^{\mu\nu}_{2e}\right]\right)$$

The 4-dimensional conformal fluid from AdS_5

Details of second order stress tensor:

$$
\begin{aligned}
T^{\mu\nu}_{2a} &= \epsilon^{\alpha\beta\gamma(\mu}\,\sigma^{\nu)}{}_{\gamma}\,u_\alpha\,\ell_\beta \\
T^{\mu\nu}_{2b} &= \sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha} - \frac{1}{3}\,P^{\mu\nu}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} \\
T^{\mu\nu}_{2c} &= \partial_\alpha u^\alpha\,\sigma^{\mu\nu} \\
T^{\mu\nu}_{2d} &= \mathcal{D}u^\mu\,\mathcal{D}u^\nu - \frac{1}{3}\,P^{\mu\nu}\,\mathcal{D}u^\alpha\,\mathcal{D}u_\alpha \\
T^{\mu\nu}_{2e} &= P^{\mu\alpha}\,P^{\nu\beta}\,\mathcal{D}\left(\partial_{(\alpha}u_{\beta)}\right) - \frac{1}{3}\,P^{\mu\nu}\,P^{\alpha\beta}\,\mathcal{D}\left(\partial_\alpha u_\beta\right) \\
\ell_\mu &= \epsilon_{\alpha\beta\gamma\mu}\,u^\alpha\partial^\beta u^\gamma,
\end{aligned}
$$

with $\mathcal{D} = u^\mu\,\partial_\mu$.

◂ $\mathcal{N} = 4$ fluid