Precision holography for non-conformal branes

Marika Taylor

University of Amsterdam

Introduction

Gravity/gauge theory duality is one of the most far reaching and important ideas to emerge in recent years.

- On the one hand it opens a window into the strong coupling dynamics of gauge theories.
- On the other hand it provides a qualitatively new paradigm for gravitational physics: spacetime is emergent, reconstructed from gauge theory data.

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- The underlying principles of holography were laid out in the foundational papers GKP, Witten: for every bulk field Φ there is a corresponding gauge invariant operator $\mathcal{O}_{\Phi}$ in the field theory, and the bulk partition function with given boundary conditions for Φ acts as the generating functional for correlation functions of this operator.
- Lifting this principle to a calculational framework requires however dealing with the divergences on both sides of the relation: the (usual) UV divergences in the field theory which are dual to IR (infinite volume) divergences in the bulk.
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References

This talk will be based mostly on

- Ingmar Kanitscheider, Kostas Skenderis and Marika Taylor
  Precision holography for non-conformal branes
  0806.xxxx

Outline

1 Non-conformal branes

2 Holographic renormalization

3 Concluding remarks

Holographic dualities

- Gravity/gauge theory dualities relate string theory in a particular background to a given non-gravitational theory (in one fewer non-compact dimensions).
- In this talk we will be interested in the low energy limit of this relation, where string theory is well approximated by classical supergravity (plus probe fundamental strings, D-branes).
- Within this limit, the duality relates the supergravity onshell action to the generating functional of connected correlators at strong 't Hooft coupling, large N.
- Precision holography requires dealing with the infinite volume divergences for given spacetime asymptotics. For AdS this is well understood, but there are many interesting gravity/gauge theory dualities outside the asymptotically AdS class...

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Non-conformal brane dualities

In this talk we will focus on non-conformal Dp-brane background/(p+1)-dimensional QFT dualities, motivated by:

1. Models for holographic QCD, such as the Witten-Sakai-Sugimoto model, which use non-conformal branes. To extract gauge theory data accurately from these models one needs precision holography.
2. More generally, one wants to explore spacetime reconstruction in cases where there is a conjectured gravity/gauge theory duality to better our understanding of holography.

Also, tests of duality against lattice results, develop of matrix theory, perhaps more potential applications (modeling transport properties in condensed matter systems at strong coupling?)

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Non-conformal branes

Supergravity solutions for fundamental strings, Dp-branes and NS5-branes can all be written in the form

$$\begin{aligned} ds^2 &= H^a(H^{-1}ds^2(E^{p,1}) + ds^2(E^{9-p})) \\ e^{\phi} &= H^b; \qquad A_{0\cdots p} = H^{-1} - 1 \end{aligned}$$

for appropriate (a, b) and H a single-centered harmonic function $H = (1 + Q/r^{7-p})$. Shortly after the AdS/CFT conjecture analogous conjectures were made (Itzhaki et al) for string theory in the decoupling limits of these backgrounds

$$ds^2 = \left(\frac{Q}{r^{(7-p)}}\right)^{a-1} ds^2(E^{p,1}) + \left(\frac{Q}{r^{(7-p)}}\right)^{a} (dr^2 + r^2 d\Omega^2_{8-p})$$

where $e^{\phi} = Q^b/r^{b(7-p)}$ and dual non-conformal $(p+1)$-dimensional theories arising as low energy limits of the worldvolume theories.

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Non-conformal branes

There are various categories which can be grouped as follows:

1. Dp-branes with $p \leq 3$. The coupling e^{ϕ} is small at the boundary of the decoupling region, and increases in the interior. The decoupling region is conformal to $AdS_{p+2} \times S^{8-p}$.
2. Dp-branes with $p = 4, 6$. The coupling e^{ϕ} diverges at the boundary of the decoupling region, but the decoupling region is still conformal to $AdS_{p+2} \times S^{8-p}$.
3. Fundamental strings. The coupling e^{ϕ} diverges at the boundary and the region is conformal to $AdS_3 \times S^8$.
4. D5 and NS5 branes. The decoupling region is a linear dilaton background with the geometry being $E^{5,1} \times R_{\rho} \times S^3$.

Here we consider only the conformally AdS cases; the 5-brane backgrounds are in a different equivalence class.

Non-conformal branes

- Note that we will initially include within the conformally AdS category fundamental strings and D6-branes.
- Whilst in both cases there is no decoupling, the conformally $AdS_{p+2} \times S^{8-p}$ geometries do solve the equations of motion - the question of holography in such a background is well posed.
- For fundamental strings, we will find that precision holography can be setup: the structure is inherited from the M2-brane and D1 case, in type IIA and IIB respectively.
- However, perhaps unsurprisingly, one cannot set up such precision holography for the D6 case.

The dual frame

In the conformally AdS cases it useful to introduce a dual frame such that $ds^2_{dual} = e^{c\phi} ds^2$ and the action becomes (Boonstra, Skenderis, Townsend):

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} e^{\gamma\phi} [R + \beta(\partial\phi)^2 - \frac{1}{2(8-p)!} |F_{8-p}|^2]$$

The field equations in this frame admit an $AdS_{p+2} \times S^{8-p}$ solution with linear dilaton.
Reducing on the sphere and (consistently) truncating to the graviton and dilaton gives the $(p+2)$-dimensional action

$$S = L \int d^{d+1}x \sqrt{-g} e^{\gamma\phi} [R + \beta(\partial\phi)^2 + C].$$

where the constants (β, γ, C) depend on the case and

$$L \sim N^{(7-p)/(5-p)} g_{YM}^{2(p-3)/(p-5)}$$

with $g^2_{YM} = g_s (\alpha')^{(p-3)/2}$ for Dp-branes and $g^2_{YM} = g_s \sqrt{\alpha'}$ for fundamental strings.

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Asymptotically AdS background

- The equations of motion admit an AdS_{d+1} solution with linear dilaton:

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\begin{aligned}
ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{dx_i dx^i}{\rho}; \\
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- Conformal invariance is broken only by the (trivial) running of the coupling.
- Thus the background admits a generalized conformal "symmetry": it is invariant under conformal transformations, provided that the string coupling g_s is transformed as a background field of appropriate weight. (Jevicki, Yoneya et al)

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Generalized conformal structure in dual theory

- Dimensional reduction of $\mathcal{N} = 1$ SYM from ten to $d = (p + 1)$ dimensions gives the bosonic terms

$$S_d = \int d^d x \sqrt{-g} \mathrm{Tr} \left(-\frac{1}{4 g_d^2} F_{ij} F^{ij} - \frac{1}{2 g_d^2} D_i X^a D^i X^a + \frac{1}{4 g_d^2} [X^a, X^b]^2 \right)$$

  where $i = 0, \cdots (d-1)$ and there are $(9 - p)$ scalars X_a. The Yang-Mills coupling g_d^2, has (length) dimension $(p - 3)$.

- Treating the coupling g_d^2 as a background field $g_d^2(x)$ transforming like a scalar field of dimension $(3 - p)$ under conformal transformations $\delta x^i = \epsilon^i$:

$$\delta g_d^2 = -\frac{1}{d}(3 - p) \nabla^i \epsilon_i g_d^2 - \epsilon^i \partial_i g_d^2,$$

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Ward identities

- Treating the coupling as a background field and requiring diffeomorphism invariance implies
$$\nabla_i T^{ij} + \mathcal{O}_{g_d^2} \partial_i g_d^2 = 0.$$
- Requiring conformal invariance of the action under the generalized conformal transformations (with the coupling transforming) implies the generalized conformal Ward identity:
$$T_i^i = (3-p) g_d^2 \mathcal{O}_{g_d^2}.$$
- Next one can ask about anomalies in these identities (assuming that supersymmetry ensures the generalized conformal structure persists at the quantum level)...

Anomalies

- In a conformal field theory, anomalies in the trace of the stress energy tensor are classified by appropriate conformal invariants, e.g. in $d = 2$

$$\langle T^i_i \rangle = cR$$

  with c the central charge and R the Ricci scalar.

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D4-brane anomaly

- Eg, for the D4-brane theory there are possible anomaly terms

$$\langle T^i_i \rangle + g^2 \langle \mathcal{O}_{g^2} \rangle = \mathcal{A} = \sum_a c_a \mathcal{A}_a$$

  where the invariants $\mathcal{A}_a$ involve both curvature and coupling, and are cubic in the curvature: $\mathcal{A}_a \sim g^2 R^3$.

- This is what one expects from the dimensional reduction of the conformal M5-brane theory, where the anomaly is

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Asymptotics

Returning to gravity, we are ready to define a linear dilaton asymptotically locally AdS structure:

We consider $(d+1)$-dimensional backgrounds which asymptotically admit generalized conformal structure, i.e. near the conformal boundary at $\rho = 0$ we can express the metric and dilaton as:

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\begin{aligned}
ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x,\rho)dx^i dx^j}{\rho}, \\
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where we expand $g(x,\rho)$ and $\kappa(x,\rho)$ in powers of ρ:

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g(x,\rho) &= g_{(0)}(x) + \rho g_{(2)}(x) + \cdots \\
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Asymptotically AdS

For $\alpha = 0$ the spacetime is asymptotically locally AdS; let us first recall how renormalization in this case works (Henningson, Skenderis, Balasubramanian, Kraus,..)

- Solving the Einstein equations perturbatively in ρ determines iteratively the coefficients in the Fefferman-Graham expansion.
- The coefficients $g_{(2n)}$ with $2n < d$ are determined entirely in terms of curvatures of $g_{(0)}$.
- For d odd, the field equations necessitate a term $g_{(d)}$ at $\rho^{d/2}$, such that only the trace of $g_{(d)}$ is determined.
- For d even, one needs to include a logarithmic term $h_{(d)}$ at order $\rho^{d/2} \ln(\rho)$ to satisfy the field equations; it is determined by $g_{(0)}$ but again only the trace of $g_{(d)}$ is determined.

Holographic renormalization

Given the asymptotic expansion, one can obtain (renormalized) holographic vevs as follows.

1. The onshell value of the gravitational action including Gibbons-Hawking boundary term

$$S = L \int d^{d+1}x\sqrt{-g}(R + C) + 2L \int d^d x\sqrt{-h}K,$$

   has infinite volume divergences, which can be regulated by evaluating at $\rho = \epsilon$.

2. These divergences in the regulated action are expressed in terms of coefficients in the asymptotic expansion.

3. Local covariant boundary counterterms S_{ct} can then be introduced to cancel the divergences and renormalize the action:

$$S_{ren} = S + S_{ct}, \qquad S_{ct} = L \int d^d x\sqrt{-h}(a_1 + a_2 R[h] + \cdots).$$

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Holographic vevs

Varying the renormalized action with respect to the source $g_{(0)ij}$ for the stress energy tensor gives the holographic vev $\langle T_{ij} \rangle$ in terms of coefficients in the asymptotic expansion.

- For d odd this results in the simple expression

$$\langle T_{ij} \rangle \sim g_{(d)ij}.$$

  Tracelessness of $g_{(d)}$ corresponds to the tracelessness of $\langle T_{ij} \rangle$ in an odd-dimensional CFT.

- For d even the expression is more complicated:

$$\langle T_{ij} \rangle \sim (g_{(d)ij} + \mathcal{T}_{ij}[g_{(0)}]),$$

  with the non-zero trace corresponding to the conformal anomaly in even dimensions.

Back to the non-conformal branes

Now let us repeat the same steps in the case with linear dilaton:
We will use the field equations to determine the terms appearing in the near boundary expansion:

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\end{aligned}
$$

for the cases of interest, parameterized by different values of (d, α, γ).

The structure depends on whether $(d - 2\alpha\gamma)/2$ is (a) an integer or (b) non-integral (cf AdS case, where $\alpha = 0$).

Almost all cases of interest fall into case (b), namely D0, D1, D2, fundamental strings, and the structure is like that of odd d AdS_{d+1} i.e.

- All coefficients $(g_{(2n)}, \kappa_{(2n)})$ for $2n < (d - 2\alpha\gamma)$ are determined in terms of $(g_{(0)}, \kappa_{(0)})$.
- One needs to include terms $(g_{(d-2\alpha\gamma)}, \kappa_{(d-2\alpha\gamma)})$ such that

$$Tr(g_{(d-2\alpha\gamma)}) + 2\kappa_{(d-2\alpha\gamma)} = 0.$$

  That the undetermined terms occur at these powers follows on dimensional grounds: they should determine the vev of the dimension d stress energy tensor, but the overall normalization of the bulk action has dimension $2\alpha\gamma$.

Case (a) includes the D3 and D4 branes, for which

- All coefficients $(g_{(2n)}, \kappa_{(2n)})$ for $2n < (d - 2\alpha\gamma)$ are determined in terms of $(g_{(0)}, \kappa_{(0)})$.
- One needs to include log terms $\rho^{(d/2-\alpha\gamma)} \ln(\rho)$ with coefficients $(h_{(d-2\alpha\gamma)}, \tilde{\kappa}_{(d-2\alpha\gamma)})$ determined in in terms of $(g_{(0)}, \kappa_{(0)})$.
- One needs to include terms $(g_{(d-2\alpha\gamma)}, \kappa_{(d-2\alpha\gamma)})$ such that

$$Tr(g_{(d-2\alpha\gamma)}) + 2\kappa_{(d-2\alpha\gamma)} = \mathcal{G}[g_{(0)}, \kappa_{(0)}].$$

Thus the structure mirrors AdS_{d+1} with d even.

Note that such asymptotic expansions are inconsistent in the D6-brane case, as anticipated.

One can now proceed to renormalization of the action as before, and from the renormalized action we can immediately compute the renormalized vev of the stess energy tensor and scalar operator...

Case (b): D0,D1,D2,F1

- The expressions for the vevs are simple in the case where $(d/2 - \alpha\gamma)$ is not an integer, with

$$\langle T_{ij} \rangle = (d - 2\alpha\gamma) L e^{\kappa_{(0)}} g_{(d-2\alpha\gamma)ij},$$

  and

$$\langle T^i_i \rangle + (p-3) g_d^2 \langle \mathcal{O}_{g_d^2} \rangle = 0.$$

- Thus the holographic stress energy tensor is determined by the appropriate term in the asymptotic expansion, and its trace is related to the vev of the scalar operator by the conformal Ward identity. Note that $\kappa_{(0)} \sim ln(g_d^2)$.
- This Ward identity is precisely that obtained using the generalized conformal structure.

Case (a): D4 brane

- The expressions for the vevs are more complicated in the case $(d/2 - \alpha\gamma)$ is an integer, with

$$\langle T_{ij} \rangle = (d - 2\alpha\gamma) L e^{\kappa_{(0)}} g_{(d-2\alpha\kappa)ij} + \mathcal{T}_{ij}[g_{(0)}, \kappa_{(0)}],$$

  and

$$\langle T^i_i \rangle + (p-3) g^2_d \langle \mathcal{O}_{g^2_d} \rangle = \mathcal{A}[g_{(0)}, \kappa_{(0)}].$$

- Thus the holographic stress energy tensor is determined non-linearly by the appropriate terms in the asymptotic expansion, and its trace is related to the vev of the scalar operator by an anomalous Ward identity.
- The anomaly descends directly from the holographic anomaly of the M5-brane theory.

D4-brane theory

- The D4-brane theory is of course non-renormalizable, because of its dimensionful coupling, so one may be surprised that the holographic "renormalization" worked so well (cf the D6-brane case).
- Recall however that the boundary condition imposed in the bulk is the asymptotic generalized conformal structure, which is in turn inherited from the dimensional reduction of the (conformal) M5-brane case.
- So fixing the asymptotics to respect this structure allows us to renormalize the theory.

Extensions to include other operators

- We discussed renormalization for the truncation to the graviton and dilaton, or equivalently the stress energy tensor and gluon operator in the field theory.
- The main advantage of top-down models like the Witten-Sakai-Sugimoto model is consistency (cf AdS/QCD): they include all gauge theory operators not just these two.
- It is straightforward to generalize analysis to linear dilaton asymptotically AdS solutions of gauged supergravities, and to the ten-dimensional linear dilaton conformally $AdS_{p+2} \times S^{8-p}$ backgrounds.
- The former allows us to include the R-currents and other low dimension scalar operators in the boundary theory; the latter involves retaining all primary operators, and is the generalization of Kaluza-Klein holography (Skenderis, Taylor 2006) to this case.

Counterterms, branes and the variational problem

- For fundamental strings (and branes) in AdS one can obtain a finite action by (a) volume renormalization by boundary counterterms (Graham-Witten, Karch et al) and (b) by adding boundary terms such that the variational problem is well-defined. (Drukker, Gross, Ooguri)
- These procedures are however equivalent (!): the counterterms are needed not just to renormalize the onshell action, but to render the variational problem well-posed.
- Finding the counterterms for brane actions is much easier than for gravity and can be done for all cases of interest, such as the D8-branes in Witten-Sakai-Sugimoto.

Examples from the Witten-Sakai-Sugimoto model

- The Witten model is based on D4-branes wrapping a circle τ with antiperiodic fermionic boundary conditions. At low (below KK scale of circle) energies the resulting non-susy theory is effectively four-dimensional QCD.
- The corresponding holographic 4-brane background is:

$$
\begin{aligned}
ds^2_{st} &= \left(\frac{U}{R}\right)^{3/2} [\eta_{ab} dx^a dx^b + f(U) d\tau^2] + \left(\frac{R}{U}\right)^{3/2} (\frac{dU^2}{f(U)} + U^2 d\Omega_4^2) \alpha'^2, \\
e^{\phi} &= g_s \left(\frac{U}{R}\right)^{3/4}, \qquad F_4 = 3N\alpha'^{3/2} d\Omega_4, \\
R^3 &= \frac{g^2_{YM} N}{\alpha'^2} = \frac{g_s N}{\alpha'^{3/2}}, \qquad f(U) = 1 - \frac{U^3_{KK}}{U^3},
\end{aligned}
$$

  Note that the dilaton ϕ couples to the dual gluon operator $\mathcal{O} = Tr(F^2)/g^2_{YM}$.
- Sakai-Sugimoto added D8-branes wrapping $E^{3,1} \times S^4$ with the embedding parameterized by a curve $\tau(U)$ to model chiral flavors in the effective four-dimensional gauge theory.

Extracting holographic data

- The vev of the gluon operator follows immediately from the holographic formulae:

$$\langle \mathcal{O} \rangle \sim \frac{1}{g_{YM}^4} U_{KK}^3$$

  The gluon condensate is in agreement with general expectations.

- One also obtains a natural structure for 2 and 3 point functions, c.f. the conformal case, e.g.:

$$\langle \mathcal{O}_\phi(x)\mathcal{O}(0)\rangle \sim N^{(7-p)/(5-p)}(g_{YM}^2)^{(p-3)/(5-p)}\mathcal{R}(x^{-(19+2p-p^2)/(5-p)}).$$

- Similarly one can extract the renormalized action of D8-branes in such backgrounds, including the finite temperature background. Such calculations tell us about the phase structure, chiral symmetry restoration etc
- The power of the formalism becomes manifest in computing those one point functions which are highly non-linear in field asymptotics, or extracting Ward identities for correlators, where naive subtraction of infinities gives inconsistent answers.

Concluding remarks

- Holography can increasingly be promoted to a very precise framework that can be used to compute field theory properties from geometry and vice versa.
- In particular, we have discussed how to decode the hologram for non-conformal brane backgrounds.
- These tools can be used to extract detailed information from phenomenological holographic models, quantified how good/bad they are, and moreover play a key role in understanding how holography works in general.
- The generalized conformal structure which underlies these theories should be fully exploited in developing these dualities further.