Degenerate conservation laws, bifurcation of solitary waves, and criticality of internal waves

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The familiar "Froude number unity" condition for criticality of a fluid of one layer in a channel is generalized to multi-layer fluids. There is a natural mapping from the velocity-depth space to energy-momentum space and it is precisely when this map is degenerate that criticality occurs. Examples of 2-layer with rigid lid, 2-layer with free surface and and 3-layer flow are used to illustrate the structure. Criticality is interesting because it signals a bifurcation to solitary waves. The more complex the basic flow, the more complex the bifurcating solitary wave. How the mechanism for creation of solitary waves is connected to criticality will be described. Generalization of criticality to non-trivial basic states will also be described.

A related context for both criticality and bifurcation of waves is degenerate conservation laws. Regularized conservation laws of the following type are of interest,

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xx} \quad \text{or} \quad \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \tag{1}$$

where $\mathbf{U} \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{D}$ is an $n \times n$ matrix. A conservation law is *degenerate* if $D\mathbf{F}(\mathbf{U}_0)$, the Jacobian, is singular for some $\mathbf{U}_0 \in \mathbb{R}^n$. In applications the flux vector is often of the type $\mathbf{F} = \mathbf{M}\nabla E(\mathbf{U})$ for an *indefinite* symmetric matrix $\mathbf{M}$ and function $E : \mathbb{R}^n \to \mathbb{R}$, with $\mathbf{M}^{-1}\mathbf{D}$ symmetric. In this case, the steady systems of (1) are gradient and Hamiltonian respectively, and in the latter case, the flux vector can be characterised as a momentum map. This structure is precisely what occurs when long-wave models are derived for multi-layer fluids.

What can one say about the *dynamics* near degeneracy of (1)? It is very similiar to criticality, but here one wants to retain both space and time. As a first step, in the case of simple degeneracy, one can derive a Burger's equation or a KdV equation that locally describes the dynamics near the direction of the eigenvector of $D\mathbf{F}(\mathbf{U}_0)$. In the case of dissipative regularisation, one can prove that the Burger's dynamics is valid. A recent proof of validity in the dissipative case, in locally uniform spaces, will be described. Higher order degeneracies, which occur in models for stratified flow, lead to coupled Burger's equations or coupled KdV equations.