Positive-existential definability in Noetherian rings

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Summary

1. (Positive-)existential sets
2. The results
3. The methods
4. Approximation properties
Conventions:

- Rings are commutative with unit.
- If $R$ is a ring, we shall consider definability over $R$ with respect to the language $L(R)$ which is the language of rings $(+, ., 0, 1)$, augmented with:

1. One constant for each element of $R$.
2. The logical constant FALSE.
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  - one constant for each element of $R$
  - the logical constant $\text{FALSE}$.
Special subsets:

If $R$ is a ring and $n \in \mathbb{N}$, then:

- **basic algebraic subsets** of $R^n$ are defined by finite systems of polynomial equations, with coefficients in $R$:

  $$\{ t \in R^n \mid F_1(t) = \cdots = F_r(t) = 0 \}$$

- **algebraic subsets** of $R^n$ are finite unions of basic algebraic subsets

- **constructible subsets** of $R^n$ are finite Boolean combinations of (basic) algebraic subsets

- **positive-existential subsets** of $R^n$ are projections of algebraic subsets of some $R^{n+p}$

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Remarks:

- One can replace “projections” by “images by polynomial maps”.
- If $R$ is a domain, all algebraic sets are basic.
- (Positive-)existential sets are those defined by (positive-)existential formulas in the language $L(R)$.
- The reason for the logical constant $\text{FALSE}$ is to make the empty subset of $R^n$ positive-existential when $R$ is the zero ring!
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- The reason for the logical constant $\text{FALSE}$ is to make the empty subset of $R^n$ positive-existential when $R$ is the zero ring!
  (for a nonzero ring, $\text{FALSE}$ is equivalent to $1 = 0$)
“Existential” vs. “positive-existential”:

Clearly, every positive-existential set is existential. The converse is true (for given $R$) if and only if $R$ is “good” in the following sense:

Definition

A ring $R$ is good if

$$R \setminus \{0\} \text{ is positive-existential in } R$$
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**Definition**

A ring $R$ is **good** if

$$R \setminus \{0\}$$

is positive-existential in $R$

and is **bad** otherwise.
Problem:

find useful classes of good (resp. bad) rings
Some examples of good rings:

- Every finite ring is good.
- Every field is good (nonzero = invertible).
- $R_1 \times R_2$ is good iff both $R_1$ and $R_2$ are.
- $\mathbb{Z}$ is good:
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  \]
- In fact, the last formula shows that every ring of algebraic integers is good.
Some examples of bad rings:

- If $p$ is a prime, $\mathbb{Z}_p$ is bad.
- More generally, infinite compact topological rings are bad (examples: $\mathbb{F}_p[[t]]$, $\mathbb{F}_p^N$).
- Infinite products of nonzero rings are bad.
- If $R$ is a nonzero ring, and $I$ is an infinite set, then $R\left((X_i)_{i \in I}\right)$ is bad.
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Some examples of bad rings:

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Main result for Noetherian domains:

“Most” Noetherian domains are good:

**Theorem**

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*Let $R$ be a Noetherian domain. If $R$ is not local Henselian, then $R$ is good.*
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What about other Noetherian rings?
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What about other Noetherian rings?

What about the Henselian case?
Other good Noetherian rings:

**Proposition**

*Let R be a Noetherian ring.*
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**Proposition**

Let $R$ be a Noetherian ring.

Assume that *every quotient domain of $R$ is good.*
Proposition

Let $R$ be a Noetherian ring.

Assume that every quotient domain of $R$ is good.

Then $R$ is good. More generally, every ring of fractions $S^{-1}R$ is good.
Corollary

*Artin rings are good.*

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*Let $R$ be a Noetherian Jacobson ring (every prime ideal is an intersection of maximal ideals).*
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Let $R$ be a Noetherian Jacobson ring.

Then *every ring of fractions $S^{-1}R$ is good.*
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Let $R$ be a Noetherian Jacobson ring.

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In particular, if $k$ is a field, every $k$-algebra essentially of finite type is good.
Local Henselian rings

Let $R$ be local with maximal ideal $\mathfrak{m}$.

Recall that $R$ is Henselian if:

Examples: complete local rings (by Hensel’s lemma) $\tilde{\mathbb{Q}} \cap \mathbb{Z}_p$. 
Local Henselian rings

Let $R$ be local with maximal ideal $m$.

Recall that $R$ is Henselian if:

for every $F \in R[X]$, every simple root of $F$ in $R/m$ lifts to a (unique) root of $F$ in $R$. 

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Examples:

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- $\overline{\mathbb{Q}} \cap \mathbb{Z}_p$. 
Local Henselian rings tend to be bad:

Assume $R$ is Noetherian, local and Henselian (not necessarily a domain). If $\text{dim } R = 0$ then $R$ is good, so we assume $\text{dim } R > 0$.

Then:

**Theorem**

If $R$ is excellent, then $R$ is bad.

**Remarks:**

All complete local rings, and all Noetherian rings "occurring naturally" in algebraic geometry and number theory, are excellent. There is a (non-excellent) Henselian discrete valuation ring which is good.
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Rings of analytic functions:

Theorem

Let $X$ be a reduced irreducible Stein analytic space.
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(e.g. a polydisk)
Rings of analytic functions:

Theorem

Let $X$ be a reduced irreducible Stein analytic space.

Then the ring $\mathcal{H}(X)$ of holomorphic functions on $X$ is good.
Some elementary facts:

- If $I$ is a finitely generated ideal of $R$ and $R/I$ is good, then $R \setminus I$ is positive-existential in $R$.

- ("Weil restriction") If some nonzero finite free $R$-algebra is good, then $R$ is good.
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The “Two Ideals” Lemma

(generalizing results of A. Shlapentokh and J. Demeyer)

Lemma

Let $R$ be a Noetherian domain, and let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of $R$. Assume that:

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**Lemma**

*Let* $R$ *be a Noetherian domain, and let* $p$ *and* $q$ *be two prime ideals of* $R$. *Assume that:*

- $p \cap q$ *contains no nonzero prime.

*Then* $R$ *is good.*
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**Lemma**

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- $\mathfrak{p} \cap \mathfrak{q}$ contains no nonzero prime
  (e.g. $\mathfrak{p}$ has height 1 and $\mathfrak{p} \not\subset \mathfrak{q}$).

Then $R$ is good.
The “Two Ideals” Lemma

Let $R$ be a Noetherian domain, and let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of $R$. Assume that:

- $\mathfrak{p} \cap \mathfrak{q}$ contains no nonzero prime.
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Explicitly, for $t \in R$, we have $t \neq 0$ if and only if

(some multiple of $t$)=$(some~x \notin \mathfrak{p})(some~y \notin \mathfrak{q})$
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(some multiple of $t$)$=(some \ x \notin \mathfrak{p})(some \ y \notin \mathfrak{q})$

and the conditions $x \notin \mathfrak{p}$ and $y \notin \mathfrak{q}$ are positive-existential.
**Corollary**

*If k is a good (Noetherian) domain, then k[X] is good.*

Proof: Apply the lemma with $R = k[X]$, $p = (X)$, $q = (X - 1)$: then

- $p \cap q = X(X - 1)R$ contains no nonzero prime,
- $R/p$ and $R/q$ are both isomorphic to $k$.

Remark: the Noetherian assumption is in fact not needed.
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Remark: the Noetherian assumption is in fact not needed.
One cannot apply the “Two Ideals” lemma directly if (for example) \( R \) is a one-dimensional local domain.

In such cases, one can try to replace \( R \) by a finite free \( R \)-algebra which has “more” primes.

For instance, if \( R = \mathbb{Z}_{(2)} \), the ring \( S = R[X]/(X^2 + X + 2) \) is free of rank 2 over \( R \) and has two maximal ideals \((X^2 + X + 2 \) has two simple roots mod 2).

The lemma then implies that \( S \) is good, hence so is \( R \) by Weil restriction.
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Note: the trick does not work if $R = \mathbb{Z}_2$
One cannot apply the “Two Ideals” lemma directly if (for example) $R$ is a one-dimensional local domain.

In such cases, one can try to replace $R$ by a finite free $R$-algebra which has “more” primes.

For instance, if $R = \mathbb{Z}_2$, the ring $S = R[X]/(X^2 + X + 2)$ is free of rank 2 over $R$ and has two maximal ideals ($X^2 + X + 2$ has two simple roots mod 2).

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Of course, this method can be used in other situations:
The “Doubling Lemma”

**Lemma**

Let $R$ be a Noetherian domain with fraction field $K$. Let $\mathfrak{p} \subset R$ be a nonzero prime ideal. Exclude the case where $R$ is local with maximal ideal $\mathfrak{p}$.

Then there exists a polynomial

$$F = X^2 + aX + b \in R[X]$$

such that $a \not\in \mathfrak{p}$, $b \in \mathfrak{p}$, and $F$ is irreducible in $K[X]$.

In particular, the $R$-algebra $S := R[X]/(F)$ has the following properties:

- $S$ is a domain,
- $S$ is free of rank 2 as an $R$-module,
- $S$ has two prime ideals above $\mathfrak{p}$, with quotients both isomorphic to $R/\mathfrak{p}$.
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The non-local case

Combining the Two Ideals Lemma, the Doubling Lemma, and an induction on dimension, one obtains:

**Proposition**

Let $R$ be a Noetherian domain, $\mathfrak{p}$ a prime ideal of $R$. Exclude the case where $R$ is local with maximal ideal $\mathfrak{p}$. If $R/\mathfrak{p}$ is good, then $R$ is good.

**Corollary**

Every *non-local* Noetherian domain is good.

Proof: apply the proposition to any maximal ideal of $R$. 
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Combining the Two Ideals Lemma, the Doubling Lemma, and an induction on dimension, one obtains:

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If $R$ is a local, non-Henselian Noetherian domain, there exists a finite $R$-algebra $S$ which is a non-local domain, hence good.
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Using Weil restriction, one concludes that $R$ is also good.
The non-Henselian case

If $R$ is a local, non-Henselian Noetherian domain, there exists a finite $R$-algebra $S$ which is a non-local domain, hence good.

Using Weil restriction, one concludes that $R$ is also good.

(Some care is needed because $S$ is not necessarily a free $R$-module).
Approximation properties and the Henselian case
Notation:

Assume

- $R$ is a ring,
- $S$ is a finite system of polynomial equations with coefficients in $R$,
- $A$ is an $R$-algebra.

Then we denote by $\text{sol}(S, A)$ the set of $A$-valued solutions of $S$. 
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Then we denote by $\text{sol}(S, A)$ the set of $A$-valued solutions of $S$. 
Let $R$ be a ring and $I$ an ideal of $R$. We say that $(R, I)$ satisfies the infinitesimal Hasse principle (IHP) if:

For each polynomial system $S$ as before,

if $\text{sol}(S, R/I^q) \neq \emptyset$ for each $q \geq 0$, then $\text{sol}(S, R) \neq \emptyset$. 
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Remarks on IHP:

Assume $R$ is local and Noetherian, with maximal ideal $I$, and $\hat{R}$ is the $I$-adic completion of $R$.

Then (IHP) is equivalent to either of:

- the approximation property: for each system $S$, $\text{sol}(S, R)$ is $I$-adically dense in $\text{sol}(S, \hat{R})$,
- the strong approximation property (Pfister-Popescu; Becker-Denef-Lipshitz-van den Dries).
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- the strong approximation property (Pfister-Popescu; Becker-Denef-Lipshitz-van den Dries).

Moreover, these properties are satisfied if $R$ is excellent (Popescu).
The connection with bad rings:

**Proposition**

Let $R$ be a Noetherian ring, $I$ an ideal of $R$. The following are equivalent:

1. $(R, I)$ satisfies the IHP,
2. for all $n$ in $\mathbb{N}$, every positive-existential subset of $R^n$ is $I$-adically closed.

(The proof is easy, directly from the definitions).
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Corollary

Let $R$ be a Noetherian ring, $I$ an ideal of $R$. Assume that $(R, I)$ satisfies the IHP and $I$ is not nilpotent. Then $R$ is bad.
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Proof: since $I$ is not nilpotent,
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Proof: since $I$ is not nilpotent, the $I$-adic topology on $R$ is not discrete.
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Let $R$ be a Noetherian ring, $I$ an ideal of $R$. Assume that $(R, I)$ satisfies the IHP and $I$ is not nilpotent. Then $R$ is bad.

Proof: since $I$ is not nilpotent, the $I$-adic topology on $R$ is not discrete. Hence, $R \setminus \{0\}$ is not closed,
Corollary

Let $R$ be a Noetherian ring, $I$ an ideal of $R$. Assume that $(R, I)$ satisfies the IHP and $I$ is not nilpotent. Then $R$ is bad.

Proof: since $I$ is not nilpotent, the $I$-adic topology on $R$ is not discrete. Hence, $R \setminus \{0\}$ is not closed, and therefore not positive-existential.
Corollary

Assume $R$ is

- Noetherian,
- local,
- Henselian,
- positive-dimensional (i.e. not Artinian),
- excellent.

Then $R$ is bad.