Young Diagrams and Concentration

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Astract. This talk presents results about concentration and fluctuations of Plancherel measure on Young diagrams with N boxes for large N.

1. Cost functions

Let $c : \mathbf{R}_+^n \times \mathbf{R}_+^n \to \mathbf{R}_+$ be the cost function

$$c(x,y) = \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{x_j + y_j} \qquad ((x_j)_{j=1}^n, (y_j)_{j=1}^n \in \mathbf{R}_+^n).$$

Say that $W : \mathbf{R}_+^n \to \mathbf{R}$ is c-convex with constant κ if there exists $\kappa > 0$ such that

$$sW(x) + (1-s)W(y) - W\big(sx + (1-s)y\big) \geq \kappa s(1-s)c(x,y)$$

for all $x, y \in \mathbf{R}_+^n$ and all $0 < s < 1$.

2. Theorem *Suppose that W is c-convex with constant κ and that*

$$\nu(dx) = Z^{-1}\exp\bigl(-W(x)\bigr)\,dx$$

is a probability measure for some constant Z. Then the transportation inequality

$$\mathrm{Tc}_c(\mu,\nu) \leq \frac{1}{\kappa}\mathrm{Ent}(\mu \mid \nu)$$

holds for all absolutely continuous probability measures μ.

3. Example Introduce the simplex

$$\Delta^n = \bigl\{x = (x_j)_{j=1}^n : n > x_1 > x_2 > \ldots > x_n > 0\bigr\},$$

and for $\beta > 0$ let

$$W(x) = \sum_{j=1}^n \log\Gamma(x_j) - \sum_{j,k:1\leq j<k\leq n} \log(x_j - x_k).$$

Then W is c-convex on Δ^n with $\kappa = 1/4$.

4. Corollary *Suppose that $g : \Delta^n \to \mathbf{R}$ is a differentiable function that satisfies*

$$\Bigl|\frac{\partial g}{\partial x_j}\Bigr| \leq L \qquad (x \in \Delta^n)$$

for some $L < \infty$ and

$$\int_{\Delta^n} g(x)\nu(dx) = 0.$$

Then

$$\int_{\Delta^n} \exp\big(tg(x)/(4L)\big)\nu(dx) \le \exp(2n^2t^2) \qquad (-1 < t < 1).$$

5. Plancherel measure on Young diagrams

Let $\lambda \vdash N$ have $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n > 0$ and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = N.$$

There are bijective correspondence between:

partitions $\lambda \vdash N$;

conjugacy classes of S_N;

irreducible characters of S_N;

irreducible representations of S_N.

Let $h_j = \lambda_j + n - j$. Then the number of standard numberings of the Young diagram associated with the partition λ is given by the hook-length formula

$$F_\lambda = \frac{N!}{\prod_{j=1}^n h_j!} \prod_{1 \le j < k \le n} (h_j - h_k).$$

The Plancherel measure on the set of irreducible representations of S_N is

$$\nu_N(\{\lambda\}) = \frac{F_\lambda^2}{N!}.$$

6. Theorem *(Vershik–Kerov, 1976)*

For each $\varepsilon > 0$,

$$\nu_N\Big\{\lambda \vdash N : \Big|\frac{\lambda_1}{\sqrt{N}} - 2\Big| > \varepsilon\Big\} \to 0 \qquad (N \to \infty).$$

- λ_1 is typically close to $2\sqrt{N}$.
- The fluctuations in λ_j are typically of order $\sqrt{\log N}$.

6. Vershik Ω distribution

Typical Young diagrams with N boxes have limiting shape given by the Ω distribution. Let

$$\begin{aligned}\Omega(x) &= \frac{2}{\pi}\Big(1 - \frac{x^2}{4}\Big)^{1/2} + \frac{x}{\pi}\sin^{-1}\frac{x}{2} - \frac{|x|}{2} \qquad (|x| < 2)\\ &= 0 \qquad (|x| > 2).\end{aligned}$$

Then the Ω distribution is the probability density function $f_\Omega : [0, 2] \to [0, 2]$ given by

$$\begin{aligned} x &= \Omega(\xi) + (1/2)\xi + (1/2)|\xi|,\\ f_\Omega(x) &= \Omega(\xi) - (1/2)\xi + (1/2)|\xi|. \qquad (7.3.5)\end{aligned}$$

6. Poisson version of Plancherel measure

Let $\Lambda = \{\lambda \vdash N; N = 0, 1, \ldots\}$ be the set of all partitions, and let $|\lambda| = N$ where $\lambda \vdash N$. For $t > 0$ let

$$P^\theta(\{\lambda\}) = e^{-\theta}\theta^{|\lambda|}\Big(\frac{F_\lambda}{|\lambda|!}\Big)^2$$

give the Poisson randomized version of Plancherel probability measure on partitions.

We extend the definition of λ_j to $\lambda(x) = \lambda_{\lceil x\rceil}$ where $\lceil x\rceil = \min\{j \in \mathbf{N} : j \geq x\}$. We introduce

$$\Delta_N^{(\lambda)}(x) = \lambda(\sqrt{N}x) - \sqrt{N}f_\Omega(x);$$

$$\theta(x) = \cos^{-1}\big((1/2)(f_\Omega(x) - x)\big)$$

for $\lambda \vdash N$ and the scaled random variables

$$Y^{(\lambda)}(x) = \frac{2\theta(x)\Delta_N^{(\lambda)}(x)}{\sqrt{\log N}} \qquad (x \in (0,2))$$

where $\lambda \vdash N$ with $\lambda \in \Lambda$ random.

7. Theorem *(Bogachev and Su, 2007)*
For $x \in (0,2)$, the distribution of $Y_t(x)$ with respect to the measure P^t converges to the $N(0,1)$ distribution as $t \to \infty$.

8. Problem What is the optimal concentration theorem for the joint distribution of row lengths?

References

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