Long-term planning versus short-term planning in location problems

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Planning an economic activity is in general an extremely complex problem, where a high number of parameters often intervene. In addition, the attitude of the planner has to be taken into account: long-term planners take their decision through an optimization process over a large time horizon, while short-term planners behave optimizing day-by-day their strategies. Usually, the first kind of behavior is perceived as more virtuous and efficient, while the second is seen as easier to implement.

The goal is to study the performances of long-term and short-term plannings on a very simple model problem: the so-called location problem, that can be roughly described as follows.

Suppose one has to open a certain given number $n \in \mathbf{N}$ of facilities (e.g. plants, shops, distribution centers, cinemas etc.) in a given urban region $\Omega \subset \mathbf{R}^d$ which for simplicity we assume to be a compact convex set.

If the density of population in Ω is described by a probability measure $\nu = f(x)dx$, and the configuration of facilities is modeled by a set $\Sigma \subset \Omega$ consisting of at most n points (i.e. $\#\Sigma \leq n$), then the simplest way to measure how good is the chosen configuration is to calculate the average distance that people have to cover to reach the nearest facility

$$F(\Sigma) := \int_\Omega \mathrm{dist}(x, \Sigma)\, f(x)\, dx.$$

If the owner of the facilities wants to open all of them at once, he would solve the problem

$$L_n = \min\Big\{F(\Sigma)\ :\ \Sigma \subset \Omega,\ \#\Sigma \le n\Big\}$$

that we call long-term location problem.

The problem is also called quantization because, given a density $f(x)$, one wants to approximate f in the best way by a discrete sum of Dirac masses, i.e. minimizing

$$W_1(f,\mu) \text{ among } \mu = \sum_{k=1}^{n} m_k \delta_{x_k}$$

being W_1 the Wasserstein distance.

The location and quantization problems have been extensively studied:
MathSciNet: “location” 3388 items in title

For instance:
Suzuki, Asami, Okabe Math. Program. ’91
Suzuki, Drezner Location Science ’96
Buttazzo, Oudet, Stepanov Birkhäuser ’02
Bouchitté, Jimenez, Rajesh CRAS ’02
Morgan, Bolton Amer. Math. Monthly ’02

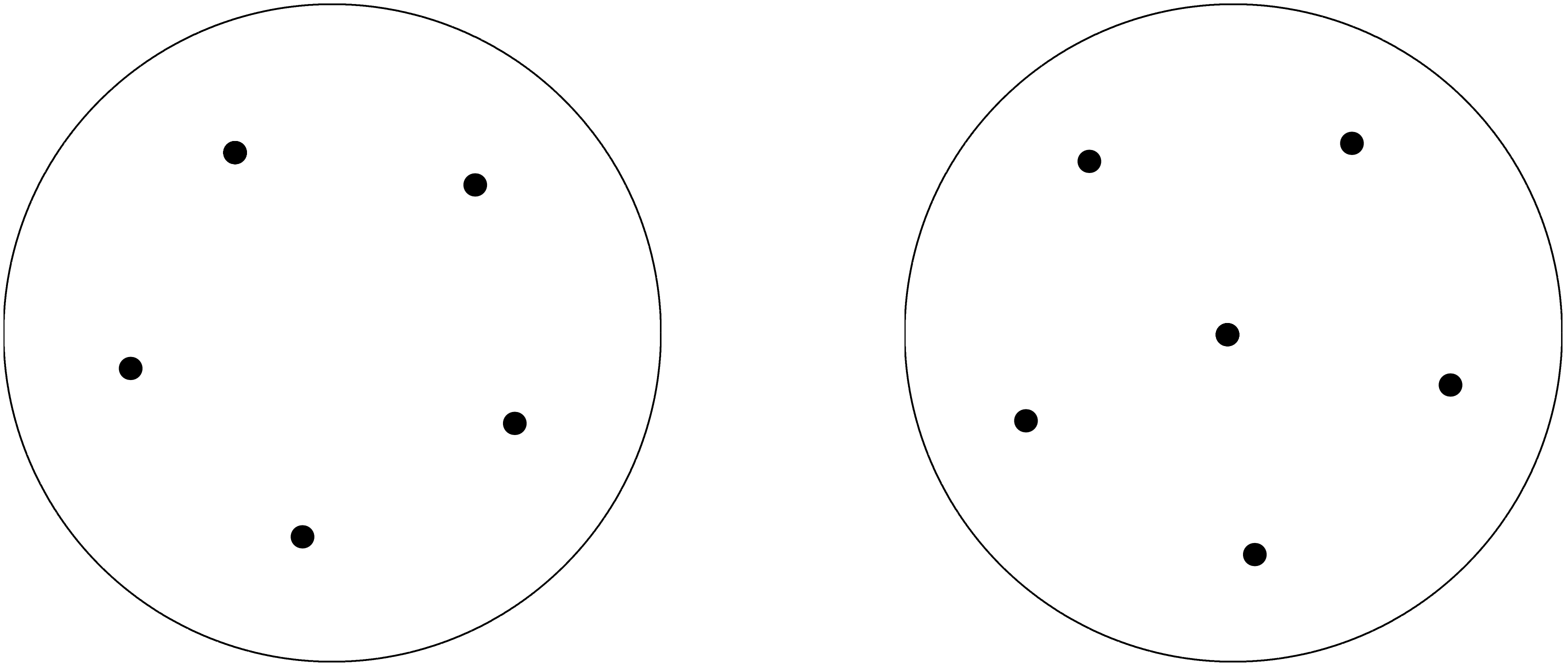
location problems in elasticity:
Buttazzo, Santambrogio, Varchon COCV ’06
Buttazzo, Santambrogio preprint ’07.

We recall here the main known facts.

- $L_n \approx n^{-1/d}$ as $n \to +\infty$;
- $n^{1/d} F_n \to C_d \int_\Omega \mu^{-1/d} f(x)\, dx$ as $n \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;

- $\mu_{opt} = K_d f^{d/(1+d)}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{d/(1+d)}$ and not as f (for instance as $f^{2/3}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by the centers of regular exagons.



Optimal locations of 5 and 6 points in a disk for $f = 1$

- In dimension one we have $C_1 = 1/4$.
- In dimension two we have

$$C_2 = \int_E |x| \, dx = \frac{3 \log 3 + 4}{6\sqrt{2}\, 3^{3/4}} \approx 0.377$$

where E is the regular hexagon of unit area centered at the origin.

- The value of the constant C_d is not known in dimension higher than two.
- In dimension higher than two the optimal asymptotical configuration of the points is not known.
- The numerical computation of optimal configurations is very heavy.

It is interesting to compare the optimal location of n points in Ω to a random distribution of n points in Ω. Surprisingly, the average cost R_n of a random location vanishes, as $n \to \infty$, with the same order of L_n, and more precisely (Cohort 2004) for $f = 1$:

$$R_n \approx R_d n^{-1/d} \qquad \text{as } n \to +\infty$$

where $R_d = \omega_d^{-1/d} \Gamma(1 + 1/d)$.

$$\begin{array}{lll} d = 1 & C_1 = 1/4 & R_1 = 1/2 \\ d = 2 & C_2 \approx 0.377 & R_2 = 1/2 \\ d = 3 & C_3 \leq 0.415 & R_3 \approx 0.554 \end{array}$$

Assume now the owner of the facilities is unable to open all the facilities immediately (say, if he does not have enough financial resources to do that); then he will open them one by one, trying to minimize the average distance functional F at each step, taking into consideration the location of facilities already opened at previous steps, i.e. solving the short-term location problem below:

$$\begin{cases} \Sigma'_0 = \emptyset \\ \Sigma'_{n+1} \text{ minimizes } F(\Sigma) \\ \qquad\quad \text{with constraints } \Sigma \supset \Sigma'_n,\ \#\Sigma \le n. \end{cases}$$

We denote by S_n the optimal cost $F(\Sigma'_n)$.

The optimal sets Σ'_n are not unique and the values S_n may depend on the choice of Σ'_k for $k < n$. We look at the questions below.

(A) Find the asymptotic order (as $n \to \infty$) of S_n , i.e. find $\alpha > 0$ such that

$$C_1 n^{-\alpha} \leq S_n \leq C_2 n^{-\alpha}$$

for some $C_1, C_2 > 0$ and for n sufficiently large. To simplify the notation, we write in this case $S_n \sim n^{-\alpha}$. Recall that, concerning the long-term problem we have

$$L_n \sim n^{-1/d}.$$

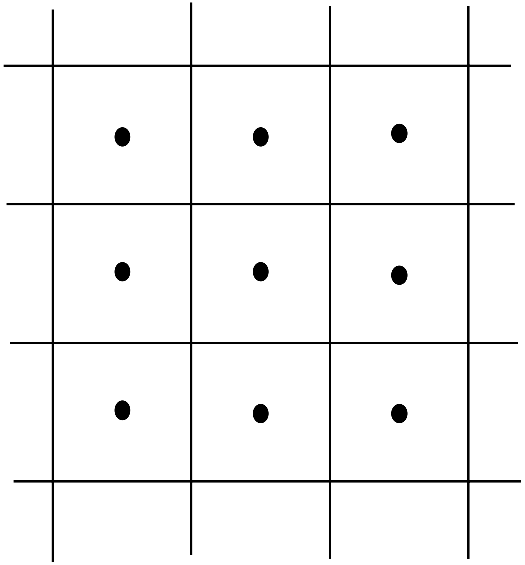
(B) Find precise asymptotic estimates for S_n, i.e. find $\lim_n n^\alpha S_n$ or just $\liminf$ and $\limsup$, should the limit not exist. Recall that in the long-term case we have

$$\lim_n n^{1/d} L_n = C_d \Big(\int_\Omega f^{d/(1+d)}\, dx \Big)^{(1+d)/d}.$$

(C) Describe the asymptotic behavior of minimizers, i.e. find all the weak limits, in a suitable sense, of subsequences of minimizers.

The answer to question (A) on the asymptotic order is very easy in the long-term case.

Estimate from above: place the n points in a regular grid of edge $\varepsilon \sim n^{-1/d}$



then we have

$$L_n \leq (\#\text{cells}) \cdot meas(\text{cell}) \cdot \varepsilon \sim n^{-1/d}.$$

Estimate from below: take a regular grid of edge ε such that the number of cells is close to $2n$, so $\varepsilon \sim n^{-1/d}$. There will be at least n empty cells so that

$$L_n \geq (\#\text{empty cells}) \cdot meas(\text{cell}) \cdot \varepsilon \sim n^{-1/d}.$$

The situation is more delicate in the short-term case.

Theorem. *We still have* $S_n \sim n^{-1/d}$*.*

Of course, since $S_n \geq L_n$, we have the estimate from below:

$$C_1 n^{-1/d} \leq L_n \leq S_n.$$

The estimate from above comes out from a careful comparison argument, through which we can prove that

$$S_{n+1} \leq S_n - K_d S_n^{d+1}.$$

The proof of the asymptotic oder estimate ends by the recursive inequality lemma below.

Lemma. Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying

$$a_{n+1} \leq a_n - C a_n^{d+1} \qquad \forall n \in \mathbf{N},$$

where $C > 0$ is a constant. Then there exists a number $B > 0$ (depending only on a_1, C and d), such that

$$a_n \leq B n^{-1/d} \qquad \forall n \in \mathbf{N}.$$

Thus question (A) is answered, and

$$L_n \sim S_n \sim n^{-1/d}.$$

We consider now questions (B) (limit problem) and (C) (asymptotic minimizers) in the one-dimensional case, with $f = 1$ for simplicity. The computation of the optimal short-term location can be made explicitly and we summarize here the results.

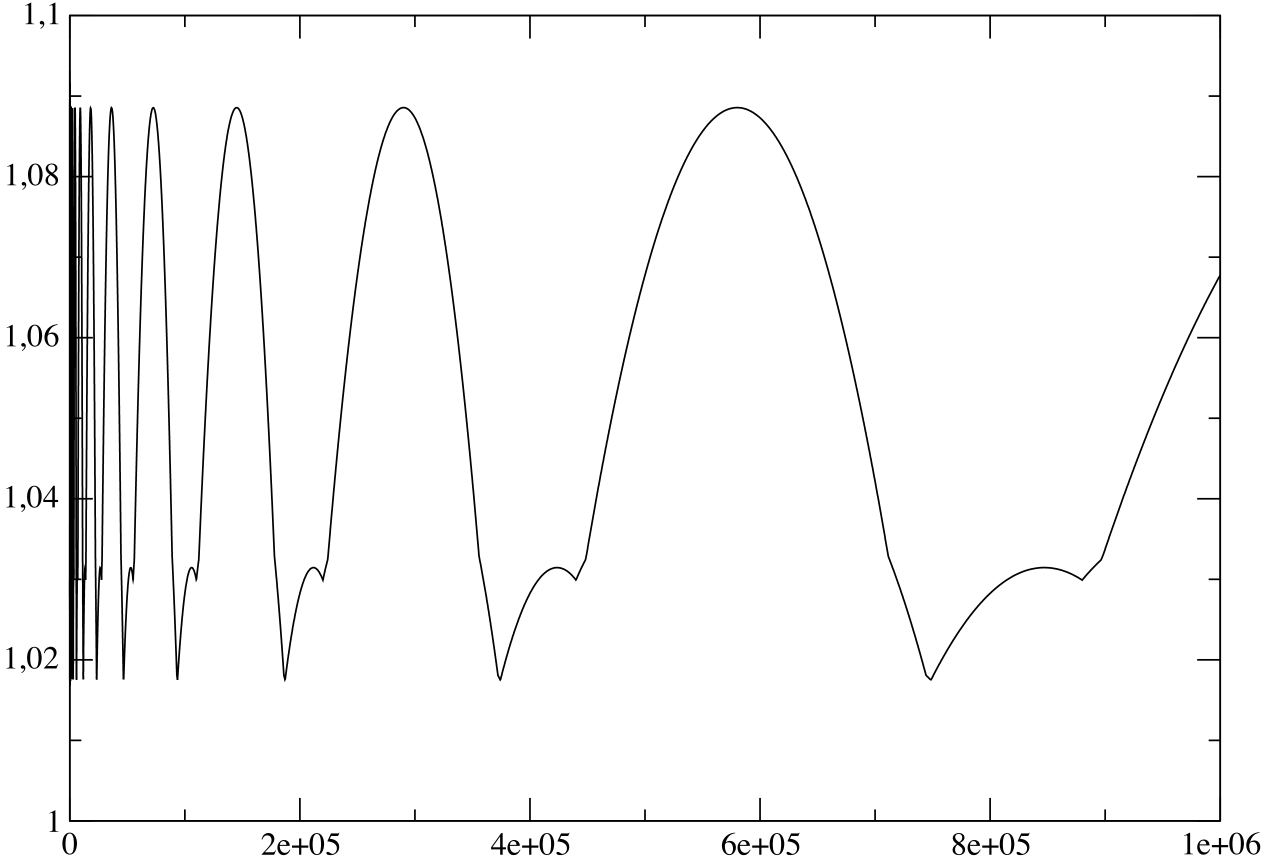
- The optimal locations Σ_n' are not unique, but in this case ($d = 1$ and $f = 1$) the optimal values S_n do not depend on the choice of Σ_n'.

- The bounded sequence nS_n does not converge to a limit as $n \to +\infty$.

- If $x_1, x_2, \ldots$ is an optimal sequence for the short-term optimization problem, then the sequence of probability measures $\mu_n = \frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n})$ has infinitely many cluster points in the weak* sense.

- No sequence Σ'_n of optimal solutions for the short-term problem has the Lebesgue measure as a cluster point in the weak* sense above.

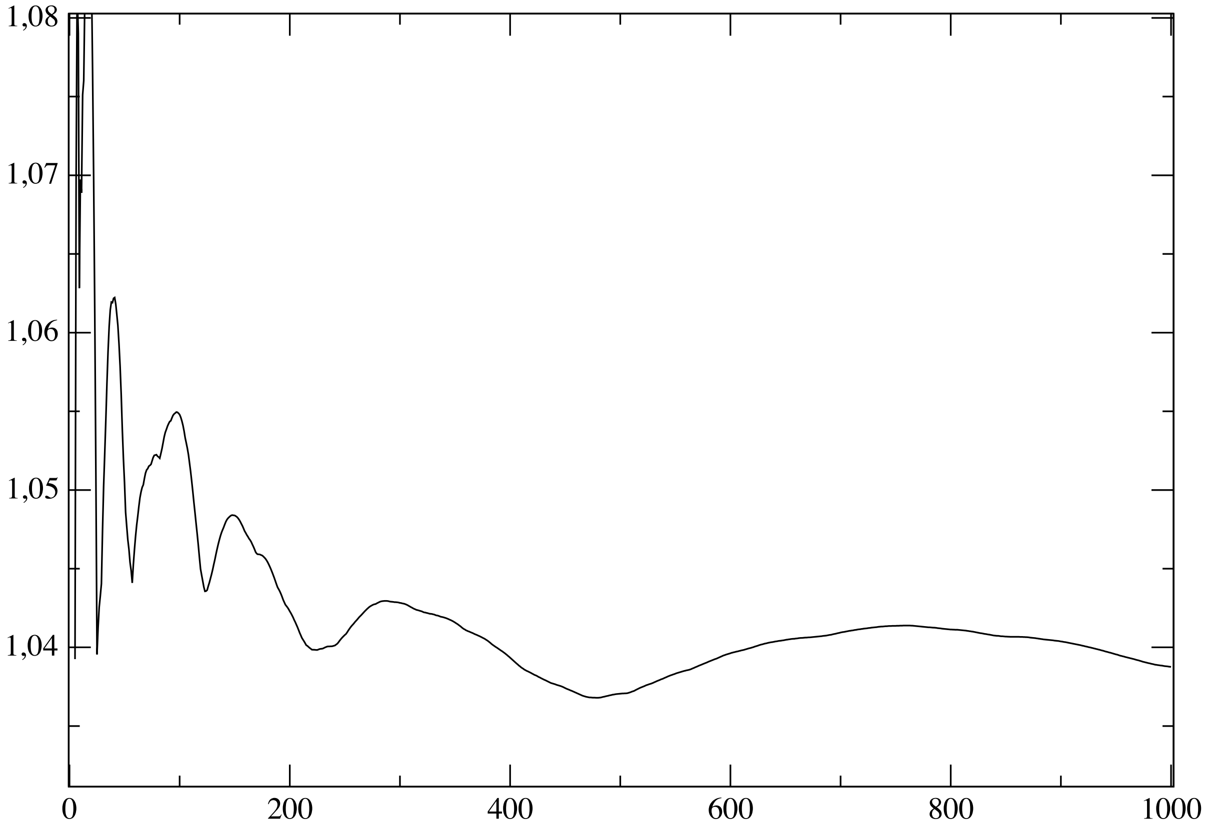
- $\liminf_{n\to\infty} S_n/L_n > 1$; nevertheless, the values of S_n and of L_n are rather close.



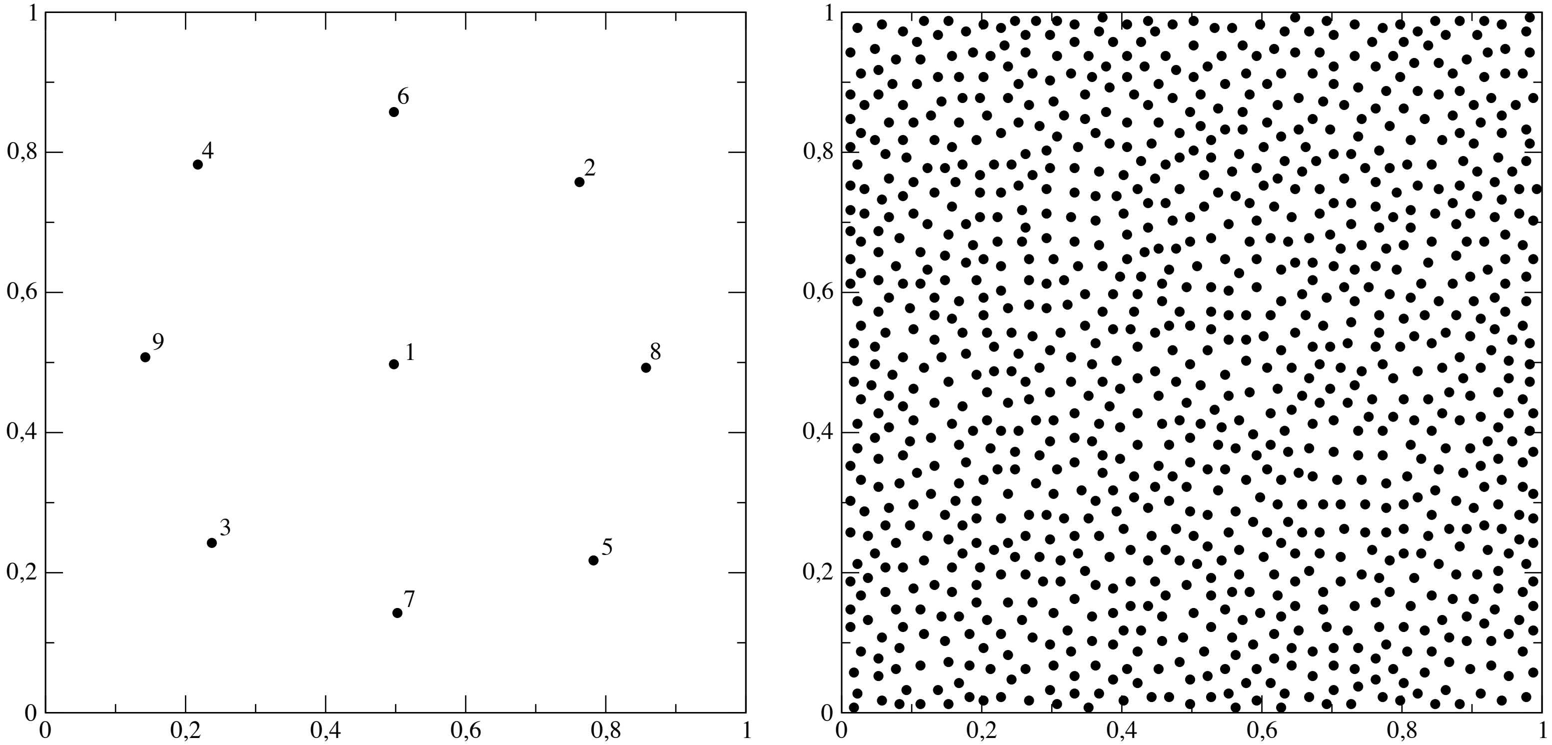
Plot of S_n/L_n for $\Omega = [0, 1]$ and $n \leq 10^6$.

We were unable to obtain similar results in dimension 2; up to now we have only questions.

- Does the limit S_n/L_n exist?
- Do we still have $\liminf_{n\to\infty} S_n/L_n > 1$?
- Is the Lebesgue measure attainable by a (sub)sequence of optimal Σ'_n ?
- Is there a characterization of all measures that can be attained by a (sub)sequence of optimal Σ'_n ?



Plot of S_n/L_n for $\Omega = [0,1] \times [0,1]$ and $n \leq 10^3$.



Plot of Σ'_n for $n = 9$ (left) and $n = 1000$ (right).

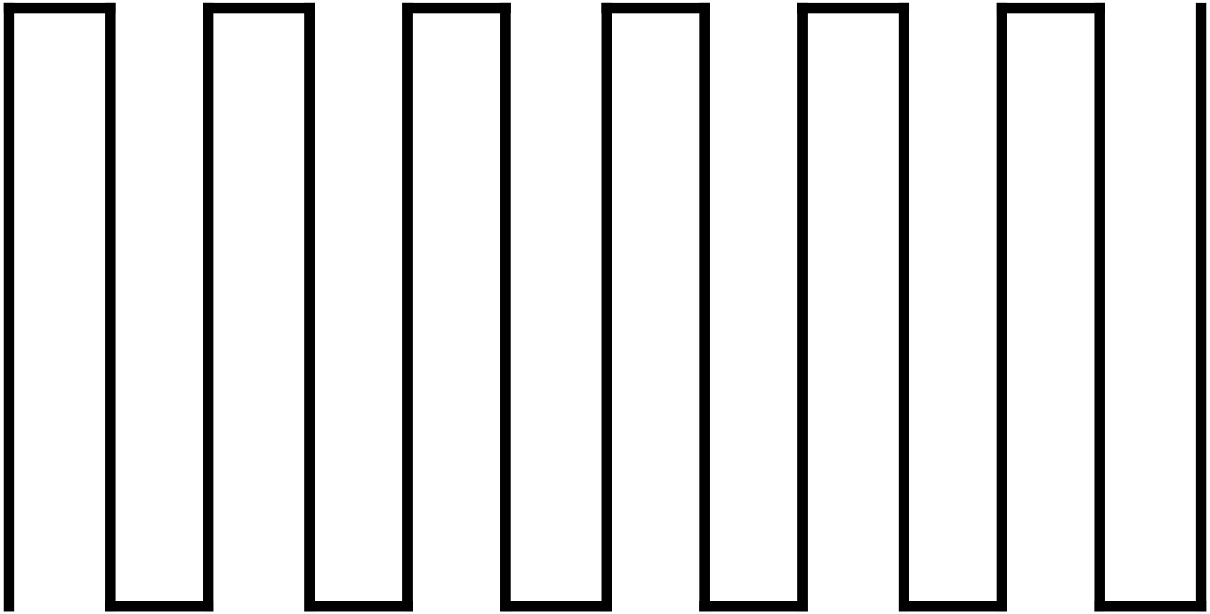
Further questions

A similar approach short-term versus long-term optimization could be done for the so-called irrigation problem where the location of n points is replaced by the location of a connected one-dimensional set of length ℓ.

The long-term analysis of the irrigation problem has been made by S.Mosconi and P.Tilli who proved the following facts.

- $L_\ell \approx \ell^{1/(1-d)}$ as $\ell \to +\infty$;
- $\ell^{1/(d-1)} F_\ell \to C_d \int_\Omega \mu^{1/(1-d)} f(x)\, dx$ as $\ell \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;
- $\mu_{opt} = K_d f^{(d-1)/d}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{(d-1)/d}$ and not as f (for instance as $f^{1/2}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by many parallel segments (at the same distance) connected by one segment.



Asymptotic long-term irrigation in dimension two.

The short-term setting for the irrigation problem can be made rigorous through a slight modification of the well-known method of the De Giorgi minimizing movements (see for instance [Ambrosio-Gigli-Savaré]) which amounts to the following.

Fixed an arbitrary time step $\tau > 0$, we start by setting $\Sigma_0^\tau = \emptyset$ and by defining Σ_{k+1}^τ as a minimizer of $F(\Sigma)$ among all connected compact subsets of $\overline{\Omega}$ satisfying the constraints

$$\Sigma_{k+1}^\tau \supset \Sigma_k^\tau \quad \text{and} \quad \mathcal{H}^1(\Sigma_{k+1}) \leq \mathcal{H}^1(\Sigma_k^\tau) + \tau.$$

Let then $t \mapsto \Sigma^\tau(t)$ be the piecewise constant map defined by

$$\Sigma^\tau(t) = \Sigma_k^\tau \qquad \text{for } t \in [k\tau, (k+1)\tau[.$$

Several questions then about the behavior of $\Sigma^\tau(t)$ arise.

- Study whether in this way one can find a well-defined increasing evolution $\Sigma(t)$ as a limit (in the Hausdorff topology) of $\Sigma^\tau(t)$ as $\tau \to 0$.

- Verify whether the quasistatic evolution $\Sigma(t)$ is made of just simple curves without any branching point.

- Study, as for the location problem, the asymptotic density of $\Sigma(t)$, i.e. the weak* limits of the measures $\mathcal{H}^1 \lfloor \Sigma(t) / \mathcal{H}^1(\Sigma(t))$ as $t \to \infty$). In particular, the questions similar to (A) about the asymptotic order of $F(\Sigma(t))$ as $t \to \infty$, to (B) about the precise asymptotic estimates, and (C) about the asymptotic behavior of minimizers, are of interest.

The results presented here can be found on

A. BRANCOLINI, G. BUTTAZZO, F. SANTAMBROGIO, E. STEPANOV: *Long-term planning versus short-term planning in the asymptotical location problem.* Preprint

available at the preprint server of our research group:

`http://cvgmt.sns.it`